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Stability and bifurcations analysis of a competition model with piecewise constant arguments

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In this paper, we investigate local and global asymptotic stability of a positive equilibrium point of system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = r_1 x(t)\left(1 - \dfrac{x(t)}{k_1}\right) - \alpha_1 x(t) y([t-1]) + \alpha_2 x(t) y([t]), \\ \dfrac{dy}{dt} = r_2 y(t)\left(1 - \dfrac{y(t)}{k_2}\right) + \alpha_1 y(t) x([t-1]) - \alpha_2 y(t) x([t]) - d_1 y(t), \end{cases}$$

where $t \geq 0$, the parameters $r_1, k_1, \alpha_1, \alpha_2, r_2, k_2$, and d_1 are positive, and $[t]$ denotes the integer part of $t \in [0, \infty)$. $x(t)$ and $y(t)$ represent population density for related species. Sufficient conditions are obtained for the local and global stability of the positive equilibrium point of the corresponding difference system. We show through numerical simulations that periodic solutions arise through Neimark–Sacker bifurcation. Copyright © 2014 John Wiley & Sons, Ltd.

Keywords: logistic equations; piecewise constant arguments; stability; bifurcation

1. Introduction

Modeling a population growth, which refers to how the number of individuals in a population increases (or decreases) with time, has a long history. One common mathematical model is the exponential growth model (or Malthusian growth model) where the growth rate is proportional to the size of the population [1]. Because exponential growth model is unrealistic, Verhulst developed a more realistic population model, namely, logistic growth model [2]. On the basis of the logistic model, Lotka and Volterra presented a more general model for competition, predation, and parasitism interactions between species [3]. In the literature, there are many versions of Lotka-Volterra models including differential equations or difference equations [1–6].

Recently, it has been developed a new concept for modeling a population growth using differential equation with piecewise constant arguments, and these equations have attracted great attention from the researchers in mathematics and biology. Differential equations with piecewise constant arguments describe hybrid dynamical systems and combine properties of both differential and difference equations and have applications in widely expanded areas such as biomedicine, chemistry, mechanical engineering, physics, civil engineering, aerodynamical engineering, and population dynamics.

In population dynamics, a first model including piecewise constant argument was constructed by Busenberg and Cooke [7] to investigate vertically transmitted diseases. Following this work, several authors have investigated the stability and oscillatory characteristics of difference solutions of logistic differential equations with piecewise constant arguments [8–18]. May [8] and May and Oster [9] have considered a simple logistic equation for a single species such as

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$$\frac{dN(t)}{dt} = rN(t)\left\{1 - \frac{N([t])}{K}\right\}, \tag{1.1}$$

where r is intrinsic growth rate and K is maximum carrying capacity. They have showed that the difference solutions of the model can be complex and chaotic for certain parameter values of r. Gopalsamy and Liu [10] have considered the differential equation

$$\frac{dN(t)}{dt} = rN(t)\{1 - aN(t) - bN([t])\}, \tag{1.2}$$

where N(t) represents the population density, r, a, and b are positive numbers, and [t] is the integer part of $t \in (0, \infty)$ The right hand side includes both regular and piecewise constant arguments, the second one estimates of the population growth performed at equally spaced time intervals. They have obtained sufficient conditions for all positive solutions of the corresponding discrete dynamic system to converge eventually to the positive equilibrium.

A more general logistic equation with piecewise constant argument

$$\frac{dx(t)}{dt} = rx(t)\left\{1 - ax(t) - b\sum_{j=0}^{m} c_j x([t-j])\right\}, \qquad t \geq 0 \tag{1.3}$$

has been investigated by Liu and Gopalsamy [11]. They have shown that for certain special cases, solutions of the equations can have chaotic behavior through period doubling bifurcations.

In modeling a population density of a bacteria species in a microcosm, Ozturk *et al.* [12] have used the differential equation

$$\frac{dx(t)}{dt} = rx(t)\{1 - \alpha x(t) - \beta_0 x([t]) - \beta_1 x([t-1])\} \tag{1.4}$$

where the parameter r is the population growth rate of the bacteria population, α, β_0, and β_1 are coefficients that each represents the irregular environmental carrying capacity for a logistic population model.

Besides the aforementioned biological models, differential equations with piecewise constant arguments have also been used for modeling tumor growth because tumor population has different dynamics properties that can be described using both differential and difference equations. For example, proliferation of the tumor cells is arranged mitosis and needs a discrete time where tumor cells have resting time and then again begin to proliferate. On the other hand, the growth and death of the population require a time-continuity. From this point of view, Bozkurt [13] has modeled an early brain tumor growth using the differential equation with piecewise constant arguments

$$\frac{dx(t)}{dt} = x(t)\{r(1 - \alpha x(t) - \beta_0 x([t]) - \beta_1 x([t-1])) + \gamma_1 x([t]) + \gamma_2 x([t-1])\} \tag{1.5}$$

and has obtained stable interval for the growth rate of tumor population, where the parameter r is the population growth rate of tumor, α, β_0, and β_1 are rates for the delayed tumor volume, γ_1 is the drug effect on the tumor, and γ_2 is a negative effect by the immune system on the tumor population.

In modeling the growth of tumor, Gatenby [19] has used the Lotka–Volterra equations as

$$\begin{cases} \dfrac{dN_1}{dt} = r_1 N_1 \left(1 - \dfrac{N_1}{k_1}\right) - \dfrac{r_1 \alpha_{12i}}{k_1} N_1 N_2 + \dfrac{r_1 \alpha_{12s}}{k_1} N_1 N_2, \\ \dfrac{dN_2}{dt} = r_2 N_2 \left(1 - \dfrac{N_2}{k_2}\right) - \dfrac{r_2 \alpha_{21}}{k_2} N_1 N_2, \end{cases} \tag{1.6}$$

where N_1 is the tumor population and N_2 is the population of normal cells from which the tumor arises. In the study of Gatenby, tumor cells compete with normal cells for space and other resources in an arbitrarily small volume of tissue within an organ.

In the present paper, we have extended model (1.6) including discrete and continuous time situations with some extra terms to study the global dynamics of the system of differential equations with piecewise constant arguments such as

$$\begin{cases} \dfrac{dx}{dt} = r_1 x(t)\left(1 - \dfrac{x(t)}{k_1}\right) - \alpha_1 x(t) y([t-1]) + \alpha_2 x(t) y([t]), \\ \dfrac{dy}{dt} = r_2 y(t)\left(1 - \dfrac{y(t)}{k_2}\right) + \alpha_1 y(t) x([t-1]) - \alpha_2 y(t) x([t]) - d_1 y(t), \end{cases} \tag{1.7}$$

where $t \geq 0$, the parameters r_1, k_1, α_1, α_2, r_2, k_2, and d_1 are positive, and [t] denotes the integer part of $t \in [0, \infty)$. x(t) and y(t) represent population density for related species. The system is based on Lotka–Volterra competition-like model that is often used in population dynamics.

The paper is organized as follows. In Section 2, we investigate discrete solutions of the system and obtain second-order discrete dynamical system. To obtain sufficient conditions for the local and global stability of the system, we use Schur–Cohn criterion and a Lyapunov function. In Section 3, we determine the Neimark–Sacker bifurcation point for the system using Schur–Cohn criterion.

2. Local and global stability analysis

We can write system (1.7) on an interval of the form $t \in [n, n+1)$ as follows

$$\begin{cases} \dfrac{dx}{dt} - \{r_1 - \alpha_1 y(n-1) + \alpha_2 y(n)\}x(t) = -r_1 K_1 (x(t))^2, \\ \dfrac{dy}{dt} - \{r_2 + \alpha_1 x(n-1) - \alpha_2 x(n) - d_1\}y(t) = -r_2 K_2 (y(t))^2, \end{cases} \tag{2.1}$$

where $\frac{1}{k_1} = K_1$, $\frac{1}{k_2} = K_2$. By solving each equation of the system (2.1), which is a Bernoulli differential equation, and letting $t \to n+1$, we obtain a system of second-order difference equations as

$$\begin{cases} x(n+1) = \dfrac{x(n)(r_1 - \alpha_1 y(n-1) + \alpha_2 y(n))}{(r_1 - \alpha_1 y(n-1) + \alpha_2 y(n) - r_1 K_1 x(n))e^{-(r_1 - \alpha_1 y(n-1) + \alpha_2 y(n))} + r_1 K_1 x(n)}, \\ y(n+1) = \dfrac{y(n)(r_2 + \alpha_1 x(n-1) - \alpha_2 x(n) - d_1)}{(r_2 + \alpha_1 x(n-1) - \alpha_2 x(n) - d_1 - r_2 K_2 y(n))e^{-(r_2 + \alpha_1 x(n-1) - \alpha_2 x(n) - d_1)} + r_2 K_2 y(n)}. \end{cases} \tag{2.2}$$

Now, we need to determine an equilibrium point to investigate global behavior of the difference system. If

$$\alpha_1 > \alpha_2, \quad r_2 > d_1 \quad \text{and} \quad r_1 > \frac{(\alpha_1 - \alpha_2)(r_2 - d_1)}{K_2 r_2}, \tag{2.3}$$

then we get a positive equilibrium point of system (2.2) such as

$$(\bar{x}, \bar{y}) = \left(\frac{K_2 r_1 r_2 + (\alpha_2 - \alpha_1)(r_2 - d_1)}{K_1 K_2 r_1 r_2 + (\alpha_1 - \alpha_2)^2}, \frac{K_1 r_1 (r_2 - d_1) + r_1(\alpha_1 - \alpha_2)}{K_1 K_2 r_1 r_2 + (\alpha_1 - \alpha_2)^2} \right). \tag{2.4}$$

The linearized system of (2.2) about $(\bar{x}, \bar{y})$ is $w(n+1) = Bw(n)$ where B is a matrix

$$B = \begin{pmatrix} e^{-K_1 r_1 \bar{x}} & 0 & \dfrac{\left(1 - e^{-K_1 r_1 \bar{x}}\right)\alpha_2}{K_1 r_1} & -\dfrac{\left(1 - e^{-K_1 r_1 \bar{x}}\right)\alpha_1}{K_1 r_1} \\ 1 & 0 & 0 & 0 \\ -\dfrac{\left(1 - e^{-K_2 r_2 \bar{y}}\right)\alpha_2}{K_2 r_2} & \dfrac{\left(1 - e^{-K_2 r_2 \bar{y}}\right)\alpha_1}{K_2 r_2} & e^{-K_2 r_2 \bar{y}} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of the matrix B is

$$\begin{aligned} p(\lambda) = \lambda^4 + \lambda^3 \left(-e^{-K_1 r_1 \bar{x}} - e^{-K_2 r_2 \bar{y}}\right) + \lambda^2 \left\{ e^{-K_2 r_2 \bar{y} - K_1 r_1 \bar{x}} + \frac{\alpha_2^2}{K_1 r_1 K_2 r_2}\left(1 - e^{-K_1 r_1 \bar{x}}\right)\left(1 - e^{-K_2 r_2 \bar{y}}\right) \right\} \\ + \lambda \left\{ \frac{-2\alpha_1 \alpha_2}{K_1 r_1 K_2 r_2}\left(1 - e^{-K_1 r_1 \bar{x}}\right)\left(1 - e^{-K_2 r_2 \bar{y}}\right) \right\} + \left(1 - e^{-K_1 r_1 \bar{x}}\right)\left(1 - e^{-K_2 r_2 \bar{y}}\right)\frac{\alpha_1^2}{K_1 r_1 K_2 r_2}. \end{aligned} \tag{2.5}$$

To determine stability conditions of discrete system, we can use the following Schur–Cohn criterion.

Theorem A ([20])
The characteristic polynomial

$$p(\lambda) = \lambda^4 + p_3 \lambda^3 + p_2 \lambda^2 + p_1 \lambda + p_0,$$

has all its roots inside the unit open disk ($|\lambda| < 1$) if and only if

(a) $p(1) > 0$ and $p(-1) > 0$,
(b) $D_1^+ = 1 + p_0 > 0$ and $D_1^- = 1 - p_0 > 0$,
(c) $D_3^+ = (1 - p_0)(1 + p_0)(p_2 + 1 + p_0) + (p_0 p_3 - p_1)(p_3 + p_1) > 0$,
(d) $D_3^- = (1 - p_0)^2(1 + p_0 - p_2) + (p_0 p_3 - p_1)(p_1 - p_3) > 0$.

Theorem 2.1
Let $(\bar{x}, \bar{y})$ the positive equilibrium point of system (2.2) and

$$2\alpha_2 < \alpha_1, \quad d_1 < r_2, \quad \frac{(\alpha_1 - \alpha_2)(r_2 - d_1)}{r_2^2} < K_2 < \frac{(\alpha_1 - \alpha_2)}{r_2}.$$

The positive equilibrium point of the system is local asymptotically stable if

$$r_2 < r_1 < \ln\left(\frac{\alpha_1}{2\alpha_2}\right) \text{ and } \frac{\alpha_1^2\left(2e^{r_1}+1\right)\left(e^{r_1}-1\right)}{r_1K_2r_2e^{2r_1}} < K_1 < \frac{\alpha_1-\alpha_2}{r_1}.$$

Proof
From characteristic equation (2.5), we can write

$$p_3 = -e^{-K_1r_1\bar{x}} - e^{-K_2r_2\bar{y}}, \quad p_2 = e^{-K_2r_2\bar{y}-K_1r_1\bar{x}} + \left(1-e^{-K_1r_1\bar{x}}\right)\left(1-e^{-K_2r_2\bar{y}}\right)\frac{\alpha_2^2}{K_1r_1K_2r_2},$$

$$p_1 = \left(1-e^{-K_1r_1\bar{x}}\right)\left(1-e^{-K_2r_2\bar{y}}\right)\frac{-2\alpha_1\alpha_2}{K_1r_1K_2r_2}, \quad p_0 = \left(1-e^{-K_1r_1\bar{x}}\right)\left(1-e^{-K_2r_2\bar{y}}\right)\frac{\alpha_1^2}{K_1r_1K_2r_2}.$$

By Theorem A(a), we have

$$p(1) = \left(1-e^{-K_1r_1\bar{x}}\right)\left(1-e^{-K_2r_2\bar{y}}\right)\left(\frac{(\alpha_1-\alpha_2)^2}{K_1r_1K_2r_2}+1\right) > 0,$$

$$p(-1) = \left(1-e^{-K_1r_1\bar{x}}\right)\left(1-e^{-K_2r_2\bar{y}}\right)\left(\frac{(\alpha_1+\alpha_2)^2}{K_1r_1K_2r_2}\right) + \left(1+e^{-K_1r_1\bar{x}}\right)\left(1+e^{-K_2r_2\bar{y}}\right) > 0.$$

It can be easily seen that (a) always holds. Analyzing Theorem A(b), we get

$$D_1^+ = 1 + \left(1-e^{-K_1r_1\bar{x}}\right)\left(1-e^{-K_2r_2\bar{y}}\right)\frac{\alpha_1^2}{K_1r_1K_2r_2} > 0, \tag{2.6}$$

$$D_1^- = 1 - \left(1-e^{-K_1r_1\bar{x}}\right)\left(1-e^{-K_2r_2\bar{y}}\right)\frac{\alpha_1^2}{K_1r_1K_2r_2} > 0. \tag{2.7}$$

It is obvious that equation (2.6) always exists. If

$$e^{K_2r_2\bar{y}} < \frac{\alpha_1^2}{\alpha_1^2 - K_1r_1K_2r_2}, \tag{2.8}$$

then (2.7) holds, where

$$K_1 < \frac{\alpha_1^2}{r_1K_2r_2}. \tag{2.9}$$

From Theorem A(c), we hold

$$\begin{aligned} D_3^+ = &\left(1+\left(1-e^{-K_1r_1\bar{x}}\right)\left(1-e^{-K_2r_2\bar{y}}\right)\frac{\alpha_1^2}{K_1r_1K_2r_2}\right)\left(1-\left(1-e^{-K_1r_1\bar{x}}\right)\left(1-e^{-K_2r_2\bar{y}}\right)\frac{\alpha_1^2}{K_1r_1K_2r_2}\right) \\ &\times\left(1+e^{-K_2r_2\bar{y}-K_1r_1\bar{x}}+\left(1-e^{-K_1r_1\bar{x}}\right)\left(1-e^{-K_2r_2\bar{y}}\right)\left(\frac{\alpha_2^2+\alpha_1^2}{K_1r_1K_2r_2}\right)\right) \\ &+\left(\left(1-e^{-K_1r_1\bar{x}}\right)\left(1-e^{-K_2r_2\bar{y}}\right)\left(\frac{\alpha_1^2e^{-K_1r_1\bar{x}}-2\alpha_1\alpha_2+\alpha_1^2e^{-K_2r_2\bar{y}}}{K_1r_1K_2r_2}\right)\right) \\ &\times\left(e^{-K_1r_1\bar{x}}+e^{-K_2r_2\bar{y}}+\left(1-e^{-K_1r_1\bar{x}}\right)\left(1-e^{-K_2r_2\bar{y}}\right)\frac{2\alpha_1\alpha_2}{K_1r_1K_2r_2}\right) > 0. \end{aligned} \tag{2.10}$$

It is shown that (2.10) holds if

$$\alpha_1^2e^{-K_1r_1\bar{x}} - 2\alpha_1\alpha_2 > 0. \tag{2.11}$$

If we consider (2.11) with $\alpha_1 > 2\alpha_2$, then we have

$$r_1 < \ln\left(\frac{\alpha_1}{2\alpha_2}\right). \tag{2.12}$$

Analyzing (d) of Theorem A, we have

$$\begin{aligned}
D_3^- = &\left(1-\left(1-e^{-K_1r_1\bar{x}}\right)\left(1-e^{-K_2r_2\bar{y}}\right)\frac{2\alpha_1^2}{K_1r_1K_2r_2}\right)\left(1-e^{-K_2r_2\bar{y}-K_1r_1\bar{x}}\right)\\
&-\left(\left(1-e^{-K_2r_2\bar{y}}\right)\frac{\alpha_1^2\left(e^{-K_1r_1\bar{x}}+e^{-K_2r_2\bar{y}}\right)-2\alpha_1\alpha_2}{K_1r_1K_2r_2}\right)\left(1-e^{-K_1r_1\bar{x}}\right)\left(e^{-K_1r_1\bar{x}}+e^{-K_2r_2\bar{y}}\right)\\
&+\left(\left(1-e^{-K_1r_1\bar{x}}\right)^2\left(1-e^{-K_2r_2\bar{y}}\right)^2\left(\frac{\alpha_1^2}{K_1r_1K_2r_2}\right)^2\right)\left(1-e^{-K_2r_2\bar{y}-K_1r_1\bar{x}}\right)\\
&+\left(1-\left(1-e^{-K_1r_1\bar{x}}\right)\left(1-e^{-K_2r_2\bar{y}}\right)\frac{\alpha_1^2}{K_1r_1K_2r_2}\right)^2\left(\left(1-e^{-K_1r_1\bar{x}}\right)\left(1-e^{-K_2r_2\bar{y}}\right)\left(\frac{\alpha_1^2-\alpha_2^2}{K_1r_1K_2r_2}\right)\right)\\
&+\left(\left(1-e^{-K_1r_1\bar{x}}\right)^2\left(1-e^{-K_2r_2\bar{y}}\right)^2\frac{\alpha_1^2\left(e^{-K_1r_1\bar{x}}+e^{-K_2r_2\bar{y}}\right)-2\alpha_1\alpha_2}{K_1r_1K_2r_2}\frac{2\alpha_1\alpha_2}{K_1r_1K_2r_2}\right)>0.
\end{aligned} \tag{2.13}$$

If

$$1-\left(1-e^{-K_1r_1\bar{x}}\right)\left(1-e^{-K_2r_2\bar{y}}\right)\frac{2\alpha_1^2}{K_1r_1K_2r_2}>0, \tag{2.14}$$

$$1-\left(1-e^{-K_1r_1\bar{x}}\right)\left(1-e^{-K_2r_2\bar{y}}\right)\frac{2\alpha_1^2}{K_1r_1K_2r_2}>\left(1-e^{-K_2r_2\bar{y}}\right)\frac{\alpha_1^2\left(e^{-K_1r_1\bar{x}}+e^{-K_2r_2\bar{y}}\right)-2\alpha_1\alpha_2}{K_1r_1K_2r_2} \tag{2.15}$$

and

$$1-e^{-K_2r_2\bar{y}-K_1r_1\bar{x}}>\left(1-e^{-K_1r_1\bar{x}}\right)\left(e^{-K_1r_1\bar{x}}+e^{-K_2r_2\bar{y}}\right), \tag{2.16}$$

then (2.13) holds. Computing (2.8), (2.14), (2.15), and (2.16) with the fact $r_1 > r_2$, we get

$$e^{K_2r_2\bar{y}}<\frac{2\alpha_1}{\sqrt{9\alpha_1^2-4K_1r_1K_2r_2}-\alpha_1}<\frac{2\alpha_1^2}{2\alpha_1^2-K_1r_1K_2r_2}<\frac{\alpha_1^2}{\alpha_1^2-K_1r_1K_2r_2}, \tag{2.17}$$

which reveal that

$$K_1>\frac{\alpha_1^2\left(2e^{r_1}+1\right)\left(e^{r_1}-1\right)}{r_1K_2r_2e^{2r_1}}, \tag{2.18}$$

and

$$K_2<\frac{\alpha_1-\alpha_2}{r_2}. \tag{2.19}$$

Under the conditions

$$\frac{(\alpha_1-\alpha_2)(r_2-d_1)}{r_2^2}<K_2<\frac{\alpha_1-\alpha_2}{r_2},$$

we can write

$$\frac{(\alpha_1-\alpha_2)(r_2-d_1)}{r_2K_2}<r_2<r_1<\ln\left[\frac{\alpha_1}{2\alpha_2}\right].$$

By (2.18) and (2.9), we have

$$\frac{\alpha_1^2\left(2e^{r_1}+1\right)\left(e^{r_1}-1\right)}{r_1K_2r_2e^{2r_1}}<K_1<\frac{(\alpha_1-\alpha_2)}{r_1}, \tag{2.20}$$

where

$$K_1<\frac{(\alpha_1-\alpha_2)}{r_1}<\frac{\alpha_1^2}{r_1K_2r_2}.$$

This completes the proof. □

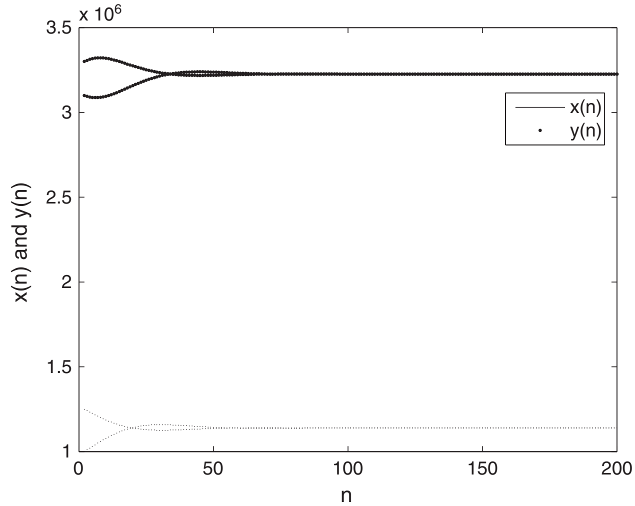

Figure 1. Graph of the iteration solution of x(n) and y(n).

Example 1
In order to determine parameter values with the biological facts, we consider a mathematical model given in [21, 22] for describing tumor and cytotoxic T lymphocytes (CTLs), which are main struggle cells of the immune system. To test the conditions of Theorem 2.1, most of the parameter values are taken from the study [21] as $r_1 = 0.18$ day^{-1}, $r_2 = 0.1045$ day^{-1}, $k_1 = 5\text{x}10^6$ cells, $k_2 = 3\text{x}10^6$ cells, $d_1 = 0.0412$ day^{-1}, $\alpha_2 = 3.422\text{x}10^{-9}$ cells^{-1}day^{-1}, and α_1 is estimated as $4.65\text{x}10^{-8}$ cells^{-1}day^{-1}. Here, r_1 and k_1 represent growth rate and carrying capacity of tumor cells respectively. r_2, k_2, and d_1 are growth rate, carrying capacity, and death rate of CTLs respectively. The parameter α_1 denotes decay rate of tumor cells by CTLs, and parameter α_2 represents decay rate of CTLs by tumor cells. Using these parameter values and initial conditions $x(1) = 1\text{x}10^6$, $x(2) = 1.25\text{x}10^6$, $y(1) = 3.1\text{x}10^6$, and $y(2) = 3.3\text{x}10^6$, the equilibrium point $(\bar{x}, \bar{y}) = (1.13938\text{x}10^6, 3.22629\text{x}10^6)$ is local asymptotically stable where x(n) and y(n) represent tumor and CTLs population density respectively (Figure 1).

Theorem 2.2
Let $\{x(n), y(n)\}_{n=-1}^{\infty}$ be a positive solution of system (2.2). Suppose that $0 < r_1 - \alpha_1 y(n-1) + \alpha_2 y(n) < 1 < r_1 K_1 x(n)$ and $0 < r_2 + \alpha_1 x(n-1) - \alpha_2 x(n) - d_1 < 1 < r_2 K_2 y(n)$ for $n = 0, 1, 2, 3 \ldots$. Then, every solution of (2.2) is bounded, that is,

$$x(n) \in \left(0, \frac{1}{r_1 K_1 (1 - e^{-1})}\right) \text{ and } y(n) \in \left(0, \frac{1}{r_2 K_2 (1 - e^{-1})}\right).$$

Proof
Because $\{x(n), y(n)\}_{n=-1}^{\infty} > 0$ and $0 < r_1 - \alpha_1 y(n-1) + \alpha_2 y(n) < 1$, it follows that

$$e^{-1} < e^{-(r_1 - \alpha_1 y(n-1) + \alpha_2 y(n))} < 1. \tag{2.21}$$

Furthermore, we have

$$-r_1 K_1 x(n) < r_1 - \alpha_1 y(n-1) + \alpha_2 y(n) - r_1 K_1 x(n) < 0. \tag{2.22}$$

Considering both (2.21) and (2.22), we get

$$\begin{aligned} x(n+1) &= \frac{x(n)(r_1 - \alpha_1 y(n-1) + \alpha_2 y(n))}{(r_1 - \alpha_1 y(n-1) + \alpha_2 y(n) - r_1 K_1 x(n)) e^{-(r_1 - \alpha_1 y(n-1) + \alpha_2 y(n))} + r_1 K_1 x(n)} \\ &< \frac{x(n)}{(r_1 - \alpha_1 y(n-1) + \alpha_2 y(n) - r_1 K_1 x(n)) e^{-(r_1 - \alpha_1 y(n-1) + \alpha_2 y(n))} + r_1 K_1 x(n)} \\ &< \frac{x(n)}{(r_1 - \alpha_1 y(n-1) + \alpha_2 y(n) - r_1 K_1 x(n)) e^{-1} + r_1 K_1 x(n)} \\ &< \frac{x(n)}{-r_1 K_1 x(n) e^{-1} + r_1 K_1 x(n)} \\ &= \frac{1}{r_1 K_1 (1 - e^{-1})}. \end{aligned}$$

Likewise, it can be shown that $y(n) \in \left(0, \frac{1}{r_2 K_2 (1 - e^{-1})}\right)$. □

Theorem 2.3
Let $\{x(n),\ y(n)\}_{n=-1}^{\infty}$ be a positive solution of system (2.2). The following statements are true.

(i) If $r_1 - \alpha_1 y(n-1) + \alpha_2 y(n) < 0$ and $r_2 + \alpha_1 x(n-1) - \alpha_2 x(n) - d_1 < 0$ for $n = 0, 1, 2, 3 \ldots$, then the solutions of system (2.2) decrease monotonically to the positive equilibrium point.
(ii) If $r_1 - \alpha_1 y(n-1) + \alpha_2 y(n) > r_1 K_1 x(n) > 0$ and $r_2 + \alpha_1 x(n-1) - \alpha_2 x(n) - d_1 > r_2 K_2 y(n) > 0$ for $n = 0, 1, 2, 3 \ldots$, then the solutions of system (2.2) increase monotonically to the positive equilibrium point.

Proof

(i) From the first equation in (2.2), we can write

$$\frac{x(n+1)}{x(n)} = \frac{A}{(A - r_1 K_1 x(n))e^{-A} + r_1 K_1 x(n)},$$

where $A = r_1 - \alpha_1 y(n-1) + \alpha_2 y(n) < 0$. Because $Ae^{-A} + r_1 K_1 x(n)(1 - e^{-A}) < 0$, we get

$$(-A + r_1 K_1 x(n))e^{-A} - r_1 K_1 x(n) + A = (e^{-A} - 1)(r_1 K_1 x(n) - A) > 0.$$

This implies that $x(n+1) < x(n)$ Similarly, It can be easily seen that if $r_2 + \alpha_1 x(n-1) - \alpha_2 x(n) - d_1 < 0$, then $y(n+1) < y(n)$.
(ii) The proof is similar with (i) and will be omitted.

□

Theorem 2.4
Let the conditions of Theorem 2.1 hold. Moreover, assume that

$$r_1 - \alpha_1 y(n-1) + \alpha_2 y(n) > 0 \ \text{ and } \ r_2 + \alpha_1 x(n-1) - \alpha_2 x(n) - d_1 > 0.$$

If

$$r_1 K_1 x(n) < r_1 - \alpha_1 y(n-1) + \alpha_2 y(n) < \ln\left(\frac{2\bar{x} - x(n)}{x(n)}\right),$$

$$r_2 K_2 y(n) < r_2 + \alpha_1 x(n-1) - \alpha_2 x(n) - d_1 < \ln\left(\frac{2\bar{y} - y(n)}{y(n)}\right),$$

and

$$x(n) < \bar{x}, \quad y(n) < \bar{y}$$

then the positive equilibrium point system (2.2) is global asymptotically stable.

Proof
Let $\bar{z} = (\bar{x}, \bar{y})$ is positive equilibrium point system (2.2), and we consider a Lyapunov function $V(n)$ defined by

$$V(n) = (Z(n) - \bar{z})^2 \qquad n = 0, 1, 2 \ldots$$

The change along the solutions of the system is

$$\begin{aligned} \Delta V(n) &= V(n+1) - V(n) \\ &= \{Z(n+1) - Z(n)\}\{Z(n+1) + Z(n) - 2\bar{z}\}. \end{aligned}$$

From the first equation in (2.2), we get

$$\begin{aligned} \Delta V_1(n) &= \{x(n+1) - x(n)\}\{x(n+1) + x(n) - 2\bar{x}\} \\ &= x(n)(A_1 - r_1 K_1 x(n))\left(1 - e^{-A_1}\right)\}\left\{A_1\left(x(n) + x(n)e^{-A_1} - 2\bar{x}e^{-A_1}\right) + r_1 K_1 x(n)\left(x(n) - 2\bar{x}\right)\left(1 - e^{-A_1}\right)\right\}, \end{aligned} \tag{2.23}$$

where $A_1 = r_1 - \alpha_1 y(n-1) + \alpha_2 y(n) > 0$.

Under the following conditions,

$$A_1 > r_1 K_1 x(n), \quad x(n) < 2\bar{x} \tag{2.24}$$

and

$$A_1 < \ln\left(\frac{2\bar{x} - x(n)}{x(n)}\right), \tag{2.25}$$

we have $\Delta V_1(n) < 0$, where $x(n) < \bar{x}$. Similarly, it can be shown that

$$\Delta V_2(n) = \{y(n+1) - y(n)\}\{y(n+1) + y(n) - 2\bar{y}\} < 0.$$

As a result, we obtain $\Delta V(n) = (\Delta V_1(n)\Delta V_2(n)) < 0$. □

Example 2

From Theorem 2.4, we can use the parameter values in Example 1 and $x(1) = 1x10^5$, $x(2) = 1x10^5$, $y(1) = 2x10^6$, $y(2) = 2x10^6$. The graph of the first 200 iterations of system (2.2) is given in Figure 2. It can be shown that under the conditions given in Theorem 2.4, the equilibrium point $(\bar{x}, \bar{y}) = (1.13938x10^6, 3.22629x10^6)$ is global asymptotically stable.

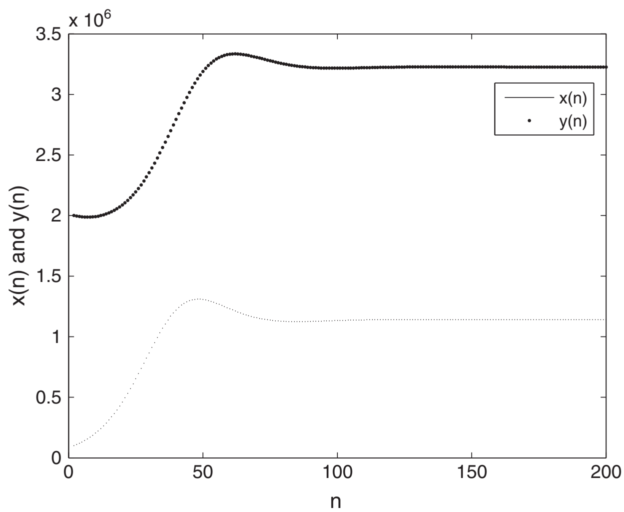

Figure 2. Graph of the iteration solution of x(n) and y(n). Parameter values are taken from Example 1.

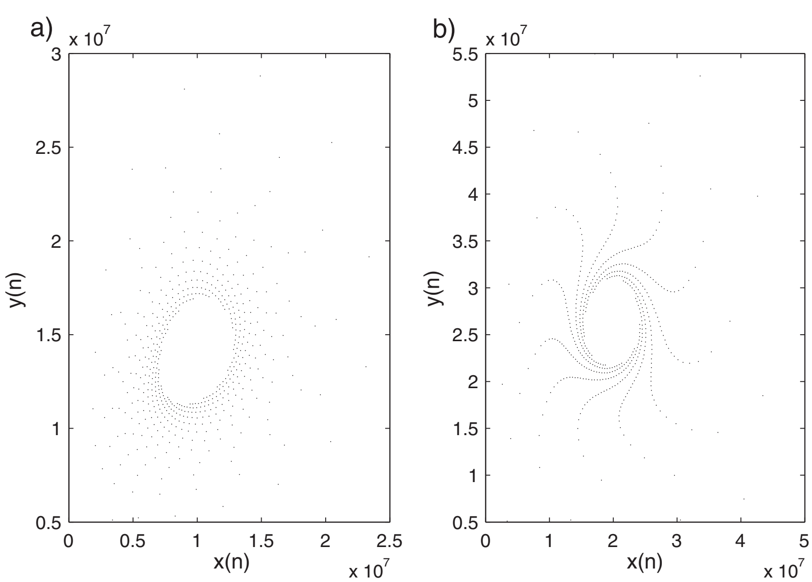

Figure 3. Graph of Neimark–Sacker bifurcation of system (2.2) for (a) $\overline{r_{11}} = 0.745271$, (b) $\overline{r_{12}} = 1.86297$, where $k_1 = 5x10^7$, $x(1) = 3.3x10^6$, $x(2) = 3.4x10^6$, $y(1) = 5x10^6$, $y(2) = 5.1x10^6$, and the other parameters are taken from Example 1.

3. Neimark–Sacker bifurcation analysis

The Neimark–Sacker bifurcation is extremely important in the context of discrete biological models because periodic or quasi-periodic solutions commonly arise as a consequence of this bifurcation of a limit cycle. For this bifurcation, characteristic equation has a pair of complex conjugate eigenvalues on the unit circle, and all other eigenvalues are inside the circle. The following theorem that is called Schur–Cohn criterion gives necessary and sufficient conditions of Neimark–Sacker bifurcation for the characteristic equation (2.5).

Theorem B ([20])
A pair of complex conjugate roots of $p(\lambda)$ lie on the unit circle and the other roots of $p(\lambda)$ all lie inside the unit circle if and only if

(a) $p(1) > 0$ and $p(-1) > 0$,
(b) $D_1^+ = 1 + p_0 > 0$ and $D_1^- = 1 - p_0 > 0$,
(c) $D_3^+ = (1 - p_0)(1 + p_0)(p_2 + 1 + p_0) + (p_0 p_3 - p_1)(p_3 + p_1) > 0$,
(d) $D_3^- = (1 - p_0)^2(1 + p_0 - p_2) + (p_0 p_3 - p_1)(p_1 - p_3) = 0$.

We have already seen that conditions of Theorem B(a) and $D_1^+ > 0$ of Theorem B(b) are always satisfied in the local stability analysis. The other conditions are analyzed numerically for different values of k_1 in the following examples.

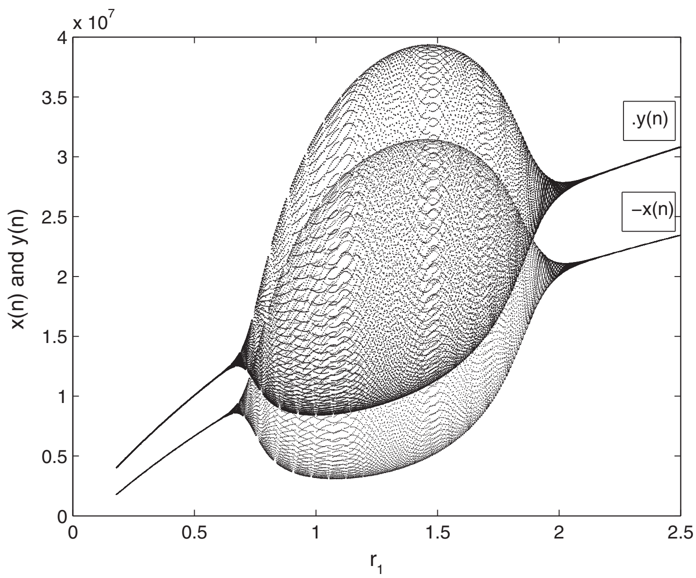

Figure 4. Bifurcation diagram of the system for $k_1 = 5x10^7$. The other parameters and initial conditions are the same as those in Figure 3.

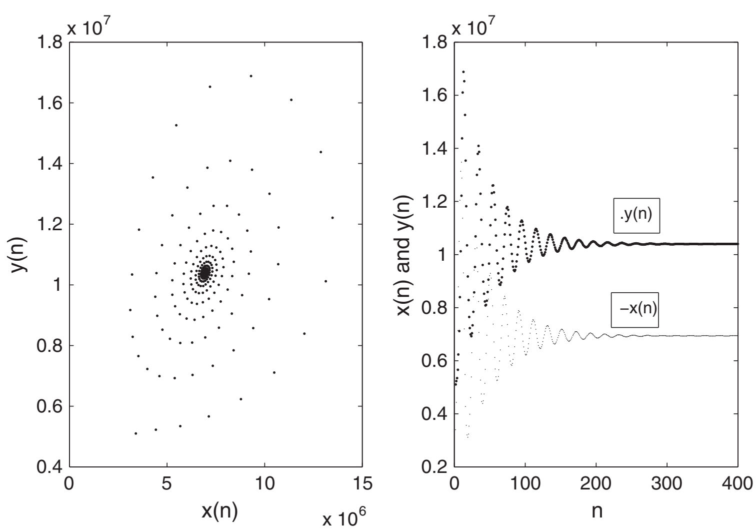

Figure 5. Graph of iteration solution of the system for $r_1 = 0.52$, where $k_1 = 5x10^7$. The other parameters and initial conditions are the same as those in Figure 3.

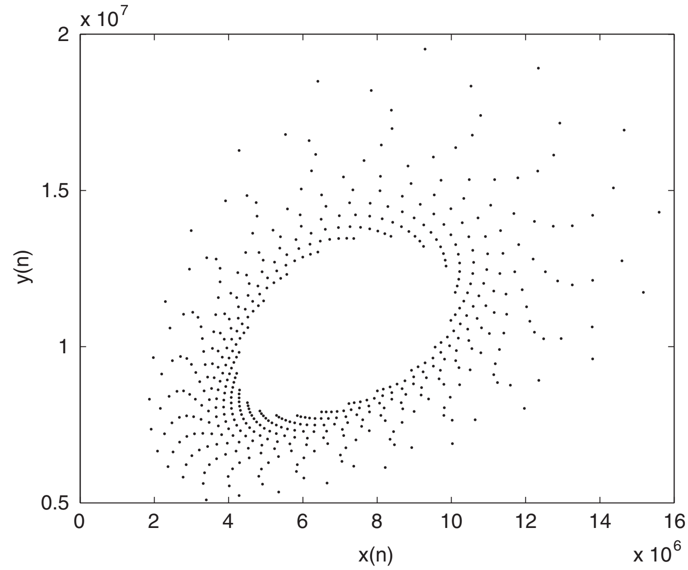

Figure 6. Graph of Neimark-Sacker bifurcation of system (2.2) for $\overline{r_1} = 0.44783$, where $k_1 = 5x10^8$. The other parameters and initial conditions are the same as those in Figure 3.

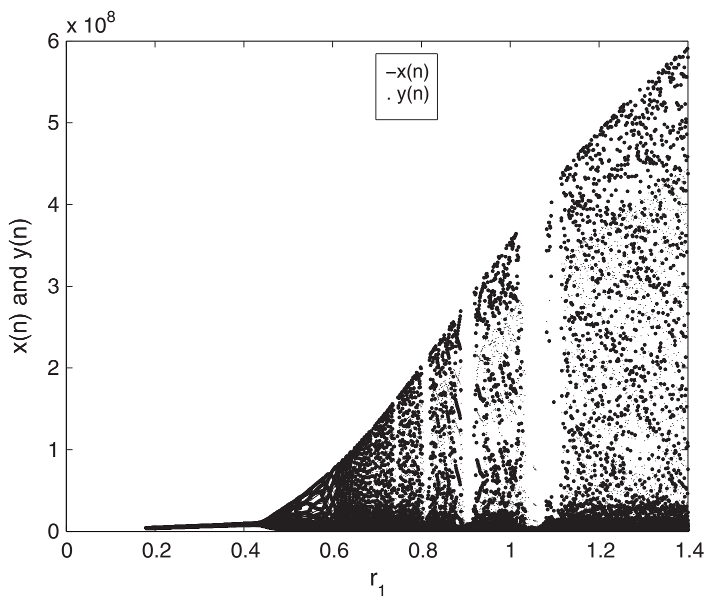

Figure 7. Bifurcation diagram of the system for $k_1 = 5x10^8$. The other parameters and initial conditions are the same as those in Figure 3.

Example 3

Solving $D_3^- = 0$ for $k_1 = 5x10^7$, we have two values of r_1, that is, $\overline{r_{11}} = 0.745271$ and $\overline{r_{12}} = 1.86297$. Furthermore, we have also $D_1^- = 0.782941 > 0$, $D_3^+ = 2.10588 > 0$ for $\overline{r_{11}}$ and $D_1^- = 0.481273 > 0$, $D_3^+ = 1.61757 > 0$ for $\overline{r_{12}}$. Figure 3a and b shows that $\overline{r_{11}}$ is the Neimark–Sacker bifurcation point of the system with the complex eigenvalues $|\lambda_{1,2}| = |-0.184778 \pm 0.427687i| = 0.465896 < 1$, $|\lambda_{3,4}| = |0.924869 \pm 0.380286i| = 1$, and $\overline{r_{12}}$ is the another Neimark-Sacker bifurcation point with the complex eigenvalues $|\lambda_{1,2}| = |-0.395606 \pm 0.601849i| = 0.720227 < 1$, $|\lambda_{3,4}| = |0.836241 \pm 0.548363i| = 1$ respectively.

The behavior of model before a Neimark–Sacker bifurcation at $r_1 = 0.52$ is shown in Figure 5. From Figure 5, we deduce that solutions of the system have damped oscillations and the positive equilibrium point is local asymptotically stable. These damped oscillations persist up to $r_1 = \overline{r_{11}} = 0.745271$. If r_1 is increased beyond this value, the norm of dominant eigenvalues of the system is greater than 1 for $0.745271 < r_1 < 1.86297$ (Figure 4). This shows that the positive equilibrium point of the system is unstable for this region. For $r_1 > \overline{r_{12}} = 1.86297$, norm of this eigenvalues again are less than 1, and the system becomes stable.

Now, we compute a bifurcation point of system for $k_1 = 5x10^8$ in the following example.

Example 4

For $k_1 = 5x10^8$, if we solve $D_3^- = 0$, then we have $\overline{r_1} = 0.44783$. Clearly, for this $\overline{r_1}$, we have also $D_1^- = 0.873197 > 0$ and $D_3^+ = 2.12931 > 0$. Figure 6 shows that $\overline{r_1}$ is the Neimark–Sacker bifurcation point of the system with the complex eigenvalues $|\lambda_{1,2}| = |-0.11228 \pm 0.337929i| = 0.356094 < 1$, $|\lambda_{3,4}| = |0.959058 \pm 0.283211i| = 1$.

After the bifurcation point, the system bifurcates to unstable situation, and no more stable dynamics or stable periodic orbits are available (Figure 7).

4. Results and discussion

In this paper, we consider a system of differential equations with piecewise constant arguments that is based on Lotka–Volterra competition-like model. Using Schur–Cohn criterion, we give some specific conditions for local asymptotic stability of the positive equilibrium point of the system in Theorem 2.1. To test these conditions, parameter values are taken from a mathematical model given in [21] for describing competition between tumor and CTLs, which are major cells of the immune system. Under the conditions of Theorem 2.1, it is observed that tumor cell $(x(n))$ and CTLs $(y(n))$ populations coexist as a stable steady state (Figure 1). Moreover, it is shown that the global stability of the system depends on initial cell density of the tumor and CTLs populations in conditions of Theorem 2.4 (Figure 2).

Because the parameter k_1 (carrying capacity of tumor population) and parameter r_1 (growth rate of tumor population) have a strong effect on the stability of the system, we choose parameter r_1 as a bifurcation parameter and investigate Neimark–Sacker bifurcation point of the system for different values of k_1. For $k_1 = 5x10^7$, bifurcation points of the system are obtained as $\overline{r_{11}} = 0.745271$ and $\overline{r_{12}} = 1.86297$ (Figures 3 and 4). In the region $r_1 < \overline{r_{11}}$, the solutions of the system have damped oscillations that give way to a stable spiral (Figure 5). If growth rate of tumor population (r_1) is increased beyond the bifurcation point, then the positive equilibrium point of system (2.2) is unstable until $r_1 = \overline{r_{12}} = 1.86297$. For $r_1 > \overline{r_{12}}$, the system again tends to stable situation as a result of competition between tumor and CTLs populations (Figure 4). On the other hand, for $k_1 = 5x10^8$, the solutions of the system exhibit different dynamic behaviors where the system bifurcates to unstable situation at the Neimark–Sacker bifurcation point $\overline{r_1} = 0.44783$ (Figure 6). If r_1 is increased beyond this value, the system has chaotic behavior, which means that the population behavior cannot be predicted (Figure 7).

As seen from the aforementioned theoretical and numerical results, the parameters r_1 and k_1 play a key role on the dynamics of the system. As the growth rate of tumor increases, tumor cells need more space and will want to increase carrying capacity (k_1). When k_1 reaches from $5x10^6$ to $5x10^7$, CTLs can suppress tumor populations, and the system tends to stable situation as a result of interaction between two populations (Figure 4). On the other hand, if k_1 reaches to $5x10^8$, CTLs cannot suppress tumor populations in the interval $r_1 > 0.44783$, and chaotic behavior occurs for the system (Figure 7).

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