Discretization of conformable fractional differential equations by a piecewise constant approximation

S. Kartal & F. Gurcan

INTERNATIONAL JOURNAL OF COMPUTER MATHEMATICS
https://doi.org/10.1080/00207160.2018.1536782

Taylor & Francis
Taylor & Francis Group

Check for updates

Discretization of conformable fractional differential equations by a piecewise constant approximation

S. Kartal[a] and F. Gurcan[b,c]

[a]Department of Science and Mathematics Education, Faculty of Education, Nevsehir Haci Bektas Veli University, Nevsehir, Turkey; [b]Department of Mathematics, Faculty of Science, Kuwait University, Safat, Kuwait; [c]Department of Mathematics, Faculty of Science, Erciyes University, Kayseri, Turkey

ABSTRACT

In this paper, a conformable fractional-order logistic differential equation including both discrete and continuous time is taken into account. By using a piecewise constant approximation, a discretization method which transforms a fractional-order differential equation into a difference equation is introduced. Necessary and sufficient conditions for both local and global stability of the discretized system are obtained. The control space diagrams (α, r) and (h, r) with the fractional-order parameter α, a discretization parameter (h) and the growth parameter (r) are obtained and these diagrams illustrate the regions where the solutions of the system approach to the positive equilibrium point with monotonic and damped oscillations. Finally, the existence of flip bifurcation is proved using the centre manifold theory and these theoretical results are supported by numerical calculations.

ARTICLE HISTORY
Received 26 May 2018
Revised 29 September 2018
Accepted 8 October 2018

KEYWORDS
Piecewise constant arguments; difference equation; stability; flip bifurcation; conformable fractional derivative

2010 AMS SUBJECT CLASSIFICATIONS
26A33; 34A08; 39A28; 39A30; 92B05

1. Introduction

The attractions of the fractional-order differential equations have a long story and attend of many researchers. One reason of this is that the theory of the fractional-order analysis is not fully developed. Another reason is that the fields of application of these equations are increasingly widening. Today, the fractional-order differential equations are used in many scientific fields such as population dynamics, fluid dynamics, mechanics, physics, epidemiology and engineering [1,5,12,19,20,24,28,29,38,44,45,50]. In population dynamics, researchers have shown that mathematical models that are established with fractional-order differential equations yield more successful results than models that are established with classical integer-order differential equations [7,11,35,39,41]. The main reason for using fractional-order differential equations is that system memory and hereditary characteristics in biological phenomena can be defined by means of these equations. Hence the next state of the fractional systems is dependent not only on its current state but also on its past (evolutionary) state.

Fractional analysis is the generalization of the classical differential and integration to the arbitrary order (non-integer state). There are many definitions of the fractional derivative such as Riemann–Liouville, Caputo and Grü nwald–Letnikov fractional derivatives.

CONTACT S. Kartal senol.kartal@nevsehir.edu.tr Department of Science and Mathematics Education, Faculty of Education, Nevsehir Haci Bektas Veli University, Nevsehir 50300, Turkey

For $\alpha \in [n-1, n)$, the Riemann–Liouville fractional derivative of f of order α is defined as

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}\,\mathrm{d}x \tag{1}$$

and Caputo fractional derivative of f of order α is

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(x)}{(t-x)^{\alpha-n+1}}\,\mathrm{d}x. \tag{2}$$

In 2014, Khalil et al. [25] first introduced a new fractional derivative namely 'conformable fractional derivative'. For all $t > 0$, $\alpha \in (0,1]$, conformable fractional derivative of $f : [0,\infty) \to R$ function is defined by

$$(T_a f)(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}. \tag{3}$$

Similarly, the left fractional derivative starting from a of the function $f : [a,\infty) \to \infty$ of order $0 < \alpha \leq 1$ is

$$(T_\alpha^a f)(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon(t-a)^{1-\alpha}) - f(t)}{\epsilon} \tag{4}$$

and the right fractional derivative of order $0 < \alpha \leq 1$ terminating at b of f is defined by

$$({}_\alpha^b Tf)(t) = -\lim_{\epsilon \to 0} \frac{f(t+\epsilon(b-t)^{1-\alpha}) - f(t)}{\epsilon}. \tag{5}$$

Note that if f is differentiable, then $(T_\alpha^a f)(t) = (t-a)^{1-\alpha} f'(t)$ and $({}_\alpha^b Tf)(t) = -(b-t)^{1-\alpha} f'(t)$ [2].

Many non-linear fractional differential equations do not have analytic solution. For this reason, approximations and numerical techniques have to be used for these equations. The numerical methods such as the Adomian decomposition [9,22,23], homotopy perturbation [26,34,36], homotopy analysis [49], variational iteration [46,48], Adams-type predictor-corrector [4,10,37,51] and differential transform method [16,17] have been used to solve the fractional differential equations recently. Grünwald–Letnikov [40,43], nonstandard finite difference scheme [6] and Euler method [8,42] are also numerical methods that are frequently used to find solution of fractional differential equations.

In recent years, a discretization process has been used to discretize the fractional order differential equations by using piecewise constant arguments [3,13,14,33]. To obtain the discretization version of the differential equation of fractional order

$$\begin{aligned} &D^\alpha x(t) = f(x(t)), \quad t > 0, \\ &x(0) = x_0, \quad t \leq 0. \end{aligned} \tag{6}$$

Agarwal et al. [3] considered the fractional order differential equations with piecewise constant arguments

$$D^\alpha x(t) = f\left(x\left(\left[\frac{t}{r}\right] r\right)\right), \quad x(t) = x_0,\ t \leq 0. \tag{7}$$

From the solutions of Equation (7) on any interval of the form $t \in [nr, (n+1)r)$, one can obtain a difference equation

$$x_{n+1}(t) = x_n(r) + \frac{(t-nr)^\alpha}{\Gamma(1+\alpha)} f(x_n(r)). \tag{8}$$

On the other hand, while adding piecewise constant arguments to the differential equations, it is important to maintain the structure of continuity in some terms from the biological point of view.

For example, while growth and death events in a population are continuous, competition of a population with another population is discontinuity. Under these biological facts, Gopalsamy and Liu [18] considered the logistic differential equations with both discrete and continuous time:

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = rN(t)(1 - aN(t) - bN([t])). \tag{9}$$

The purpose of the present paper is to study dynamic behaviour of fractional order version of Equation (9) such as

$$T_\alpha^a N(t) = rN(t)\left(1 - aN(t) - bN\left(\left[\frac{t}{h}\right]h\right)\right). \tag{10}$$

2. Discretization process

Consider the fractional order logistic equation with piecewise constant argument given by

$$T_\alpha^a N(t) = rN(t)\left(1 - aN(t) - bN\left(\left[\frac{t}{h}\right]h\right)\right) \tag{11}$$

with the initial condition $N(0) = N_0$, where $[t]$ denotes the integer part of $t \in [0, \infty)$ and $h > 0$ is a discretization parameter. Let $t \in [nh, (n+1)h),\ n = 0, 1, 2, \ldots$. By using the left conformable fractional derivative, we have

$$(t - nh)^{1-\alpha}\frac{\mathrm{d}N(t)}{\mathrm{d}t} = rN(t)(1 - aN(t) - bN(nh)), \tag{12}$$

which leads to

$$-\frac{N'(t)}{N^2(t)} + \frac{(r - rbN(nh))}{(t - nh)^{1-\alpha}}\frac{1}{N(t)} = \frac{ra}{(t - nh)^{1-\alpha}}. \tag{13}$$

If we multiply both sides of Equation (13) by $\mathrm{e}^{(r - rbN(nh))((t-nh)^\alpha/\alpha)}$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{N(t)}\,\mathrm{e}^{(r-rbN(nh))((t-nh)^\alpha/\alpha)}\right) = \frac{ra}{(t - nh)^{1-\alpha}}\,\mathrm{e}^{(r-rbN(nh))((t-nh)^\alpha/\alpha)}, \quad t \in [nh, (n+1)h). \tag{14}$$

Integrating both sides of (14) with respect to t on $[nh, t)$, one obtains

$$\begin{aligned}&\frac{1}{N(t)}\,\mathrm{e}^{(r-rbN(nh))((t-nh)^\alpha/\alpha)} - \frac{1}{N(nh)}\,\mathrm{e}^{(r-rbN(nh))((nh-nh)^\alpha/\alpha)} \\ &\quad = \frac{a}{1 - bN(nh)}(\mathrm{e}^{(r-rbN(nh))((t-nh)^\alpha/\alpha)} - \mathrm{e}^{(r-rbN(nh))((nh-nh)^\alpha/\alpha)}).\end{aligned} \tag{15}$$

We let $t \to (n+1)h$ in (15) and obtain

$$N((n+1)h) = \frac{N(nh)(1 - bN(nh))}{(1 - bN(nh) - aN(nh))\,\mathrm{e}^{-r(1-bN(nh))(h^\alpha/\alpha)} + aN(nh)}. \tag{16}$$

Adapting the difference equation notation and replacing $N(nh)$ by $N(n)$ gives

$$N(n+1) = \frac{N(n)(1 - bN(n))}{(1 - bN(n) - aN(n))\,\mathrm{e}^{-r(1-bN(n))(h^\alpha/\alpha)} + aN(n)}, \quad n = 0, 1, 2, \ldots. \tag{17}$$

If we let $bN(n) \equiv x(n)$, then Equation (17) reduces to

$$x(n+1) = \frac{x(n)(1-x(n))}{(1-x(n)-cx(n))\,\mathrm{e}^{-r(1-x(n))(h^{\alpha}/\alpha)} + cx(n)} = f(x,r), \quad n = 0,1,2,\ldots. \tag{18}$$

where $c = a/b$.

3. Local and global stability analysis

In this section, we study the stability of the equilibrium points of Equation (18) which has two equilibrium points namely, $x^* = 0$ and $x^* = 1/(1+c)$. The following theorem gives the necessary and sufficient condition for the local asymptotically stable of the positive equilibrium point $x^* = 1/(1+c)$.

Theorem 3.1: *Assume that $c \in [0,1)$. The positive equilibrium point of system (18) is local asymptotically stable if and only if*

$$r < \frac{\alpha}{h^{\alpha}} \frac{1+c}{c} \ln\left(\frac{1+c}{1-c}\right). \tag{19}$$

Proof: The eigenvalue of the linear equation of (18) at the positive equilibrium point is

$$\lambda = \left.\frac{\partial f}{\partial x}\right|_{x^*=1/(1+c)} = \frac{1}{c}(\mathrm{e}^{-(cr/(1+c))(h^{\alpha}/\alpha)}(1+c)-1). \tag{20}$$

The asymptotically stable condition $|\lambda| < 1$ leads to

$$\frac{1-c}{1+c} < \mathrm{e}^{-(cr/(1+c))(h^{\alpha}/\alpha)} < 1. \tag{21}$$

It can be easily seen that if $c > 1$, then the inequality (21) holds for all $r \in [0,\infty)$. Otherwise, if $c \in [0,1)$, then we have

$$r < \frac{\alpha}{h^{\alpha}} \frac{1+c}{c} \ln\left(\frac{1+c}{1-c}\right), \tag{22}$$

which completes the proof. ∎

The following theorem gives a region where the solutions of Equation (18) approach to positive equilibrium point with monotonic and damped oscillations.

Theorem 3.2:

(a) *The eventual convergence of solutions to x^* is monotonic (nonoscillatory) if*

$$r < \frac{\alpha}{h^{\alpha}} \frac{1+c}{c} \ln(1+c). \tag{23}$$

(b) *The eventual convergence of solutions to x^* is damped oscillatory if*

$$\frac{\alpha}{h^{\alpha}} \frac{1+c}{c} \ln(1+c) < r < \frac{\alpha}{h^{\alpha}} \frac{1+c}{c} \ln\left(\frac{1+c}{1-c}\right). \tag{24}$$

Proof: (a) We note that the eventual convergence of solutions to x^* is monotonic if $0 < \lambda < 1$. Therefore from Equation (20), we have

$$\frac{1}{1+c} < \mathrm{e}^{-(cr/(1+c))(h^\alpha/\alpha)} < 1, \tag{25}$$

which gives

$$r < \frac{\alpha}{h^\alpha}\frac{1+c}{c}\ln(1+c). \tag{26}$$

(b) If $-1 < \lambda < 0$, then the eventual convergence of solutions to x^* is damped oscillatory. Now, we have

$$\frac{1-c}{1+c} < \mathrm{e}^{-(cr/(1+c))(h^\alpha/\alpha)} < \frac{1}{1+c}, \tag{27}$$

that is

$$\frac{\alpha}{h^\alpha}\frac{1+c}{c}\ln(1+c) < r < \frac{\alpha}{h^\alpha}\frac{1+c}{c}\ln\left(\frac{1+c}{1-c}\right). \tag{28}$$

■

Example 3.3: Figure 1 shows a control space diagram (α, r) which is divided into three regions. In the lower and middle regions, the solutions are eventually monotonic and damped oscillatory (stable) so are convergent to the positive equilibrium point x^* respectively. In the upper region, the solutions of Equation (18) do not approach to positive equilibrium point (unstable). This figure shows that as the parameter α is increasing from 0 to 1 the stability region is linearly increasing. For example, to determine the value of r, we substitute $c = 0.25$, $h = 1$ and $\alpha = 1$ into the inequalities (19), (23) and (24), thus the monotonic, damped oscillatory and unstable regions can be obtained as $r < 1.11572$, $1.11572 < r < 2.55413$ and $r > 2.55413$ respectively (see Figure 1). Figure 2 shows also a control space diagram (h, r) which shows the effect of discretization parameter (h) on the dynamic behaviour of the equation. Similar type of regions can be observed as in Figure 1 but the shape of the region is different as the parameters (r and h) are changed. It is found that as the parameter h is increasing from 0 to 1 the stability region is linearly decreasing exponentially. For $c = 0.25$, $h = 0.4$, $\alpha = 1$, monotonic, damped oscillatory and unstable regions are obtained as $r < 2.78929$, $2.78929 < r < 6.38532$ and $r > 6.38532$, respectively.

Theorem 3.4: *Suppose that $c \in [1, \infty)$, $r \in [0, \infty)$. Then the positive equilibrium point $x^* = 1/(1+c)$ of Equation* (18) *is globally asymptotically stable.*

Proof: We consider a Lyapunov function V defined by $V(n) = (x(n) - x^*)^2$, $n = 0, 1, 2, \ldots$. The change along the solutions of the system is

$$\begin{aligned}\Delta V(n) = V(n+1) - V(n) &= (x(n+1) - x^*)^2 - (x(n) - x^*)^2 \\ &= (x(n+1) + x(n) - 2x^*)(x(n+1) - x(n)).\end{aligned} \tag{29}$$

Considering Equation (18), we get

$$x(n+1) - x(n) = \frac{x(n)}{F(1-x(n))}\left(\mathrm{e}^{r(1-x(n))(h^\alpha/\alpha)} - 1\right)(x^* - x(n))(1+c) \tag{30}$$

and

$$x(n+1) + x(n) - 2x^* = \frac{x(n)}{F}\,\mathrm{e}^{r(1-x(n))(h^\alpha/\alpha)} + x(n) - 2x^*$$

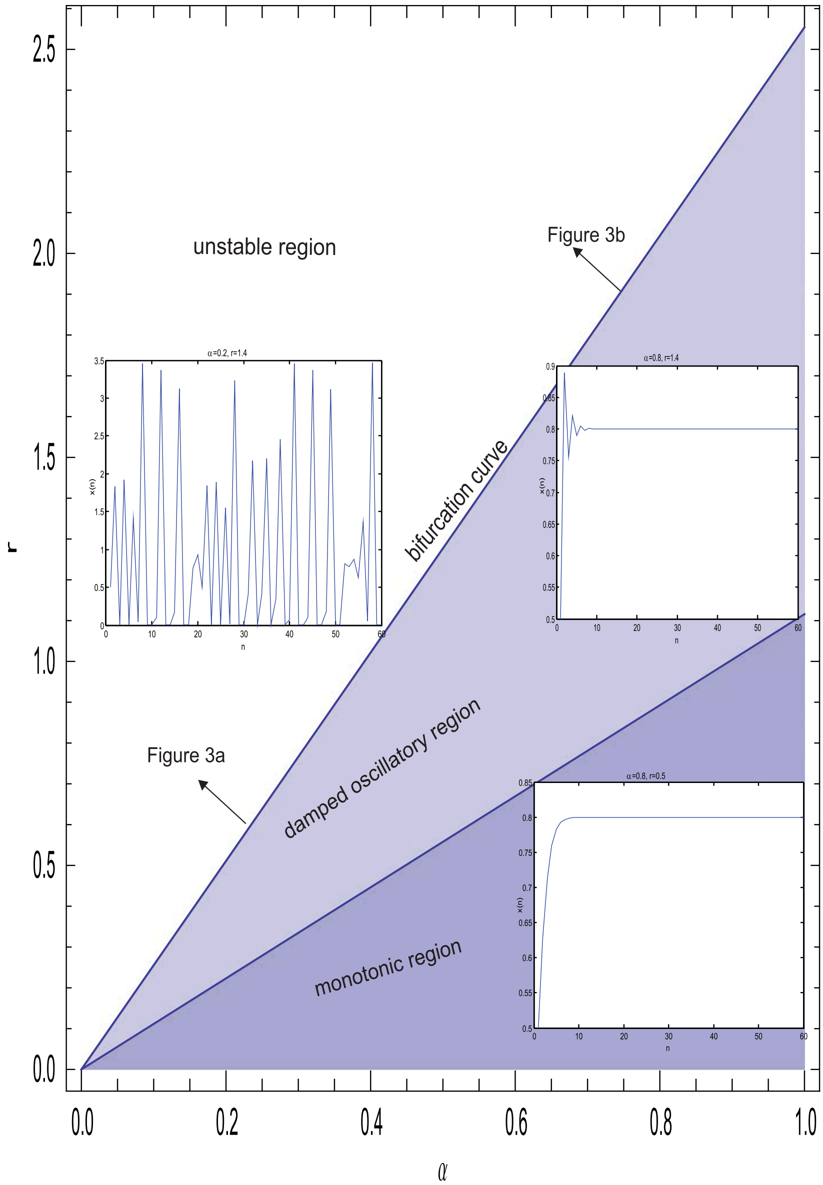

Figure 1. Control space diagram (α, r) shows regions in different types of the stability with $c = 0.25$, $h = 1$.

$$= \frac{x(n)}{F(1-x(n))}(e^{r(1-x(n))(h^{\alpha}/\alpha)} - 1)(x^* - x(n))(1+c)$$
$$\times \left(\frac{1-c}{1+c} - \frac{2}{(1+c)x(n)} \left(\frac{1-x(n)}{e^{r(1-x(n))(h^{\alpha}/\alpha)} - 1} \right) \right), \tag{31}$$

where

$$F = 1 + cx(n) \left(\frac{e^{r(1-x(n))(h^{\alpha}/\alpha)} - 1}{1-x(n)} \right). \tag{32}$$

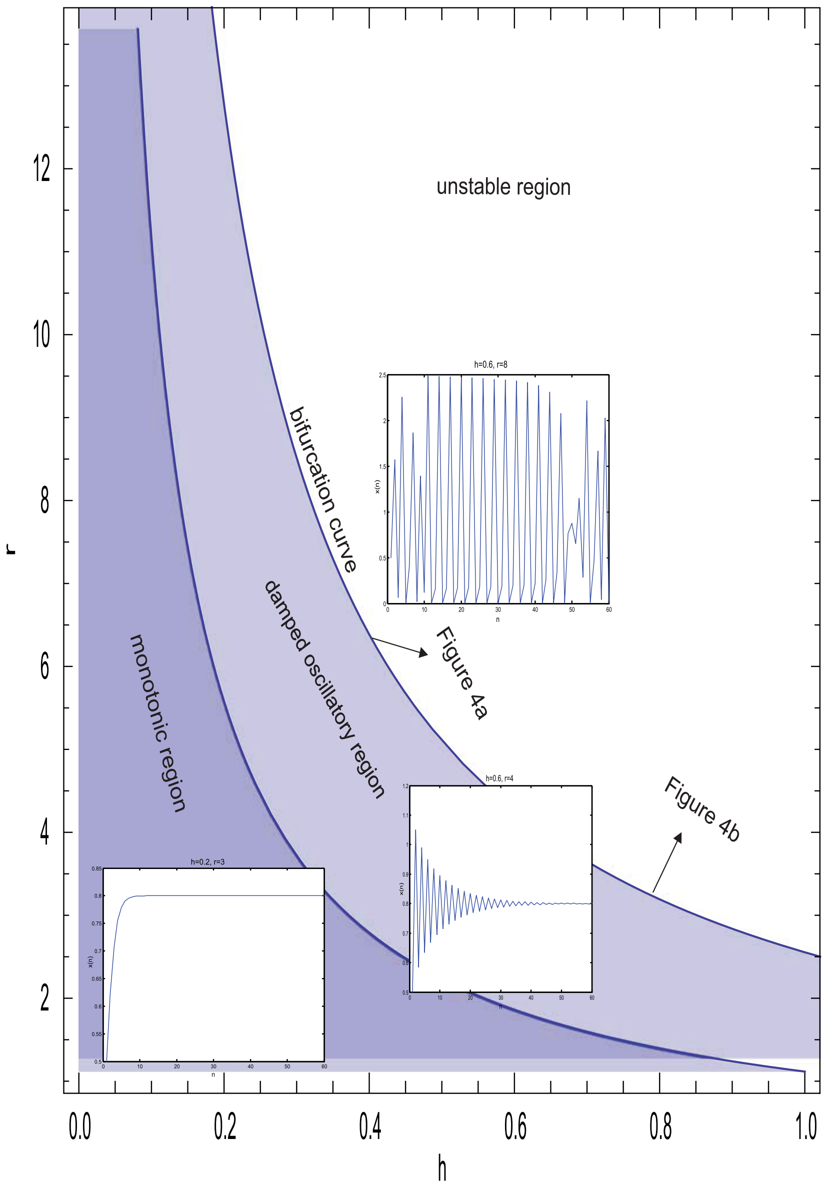

Figure 2. Control space diagram (h, r) shows regions in different types of the stability with $c = 0.25, \alpha = 1$.

From the inequalities (30) and (31), we have

$$\begin{aligned}\Delta V(n) &= (x(n+1) + x(n) - 2x^*)(x(n+1) - x(n)) \\ &= \left(\frac{x(n)}{F(1-x(n))}(\mathrm{e}^{r(1-x(n))(h^\alpha/\alpha)} - 1)(x^* - x(n))(1+c)\right)^2 \\ &\quad \times \left(\frac{1-c}{1+c} - \frac{2}{(1+c)x(n)}\left(\frac{1-x(n)}{\mathrm{e}^{r(1-x(n))(h^\alpha/\alpha)} - 1}\right)\right) \le 0, \quad \text{for } c \ge 1, \qquad (33)\end{aligned}$$

which completes the proof. ■

4. Bifurcation analysis

In this section, we will study direction and stability of the Flip bifurcation in (18) [27,47]. To study Flip bifurcation, the parameter r is chosen as a bifurcation parameter. Now, we can investigate the conditions and direction of Flip bifurcation.

Theorem 4.1: *Suppose that x^* is the positive equilibrium point of Equation (18). If $c \neq 1$ and $c \neq \frac{1}{2}\ln((1+c)/(1-c))$ then Equation (18) undergoes a flip bifurcation at the equilibrium point x^* when the parameter r varies in a small neighbourhood of r_1.*

Proof: The eigenvalue of the linear equation of (18) at the positive equilibrium point $x^* = 1/(1+c)$ is

$$\lambda(r) = f_x(x, r) = \frac{1}{c}(\mathrm{e}^{-(cr/(1+c))(h^\alpha/\alpha)}(1+c) - 1). \tag{34}$$

On the other hand, the condition $\lambda = -1$ gives the critical flip bifurcation point r_1 as

$$r = r_1 = \frac{\alpha}{h^\alpha}\frac{1+c}{c}\ln\left(\frac{1+c}{1-c}\right). \tag{35}$$

From the transversality conditions, we get

$$f_{x\alpha}(x^*, r_1) = \frac{h^\alpha}{\alpha}\frac{c-1}{1+c} \neq 0, \quad \text{for } c \neq 1. \tag{36}$$

In addition, we have

$$f_{xx}(x^*, r_1) = \frac{2(c^2-1)(2c - \ln(\frac{1+c}{1-c}))}{c^2} \tag{37}$$

and

$$f_{xxx}(x^*, r_1) = \frac{3(c-1)(1+c)^2(2c - \ln(\frac{1+c}{1-c}))(-2+4c-\ln(\frac{1+c}{1-c}))}{c^3}. \tag{38}$$

Now, the non-degeneracy condition can be calculated as

$$\begin{aligned} d(0) &= \frac{1}{4}(f_{xx}(x^*, r_1))^2 + \frac{1}{6}f_{xxx}(x^*, r_1) \\ &= \frac{(c-1)(1+c)^2(2c - \ln(\frac{1+c}{1-c}))(2c + (-2+c)\ln(\frac{1+c}{1-c}))}{2c^4} \neq 0. \end{aligned} \tag{39}$$

■

Now, we present the bifurcation diagrams, phase portraits for the system to confirm the above theoretical analysis and show the complex dynamical behaviours by using numerical simulations.

Example 4.2: For the parameters values $c = 0.25$, $h = 1$, $\alpha = 0.2$, the critical value of Flip bifurcation point is obtained from Equation (35) as $r_1 = 0.510826$ (Figure 3 a). For this value, it is easy to see that $\lambda(r_1) = -1$. On the other hand, from Equations (36) and (39), we have $f_{x\alpha}(x^*, r_1) = -3$ and $d(0) = 0.639705$ which show that a unique and stable period-two cycle bifurcates from x^* for $r < r_1 = 0.510826$. For $\alpha = 0.8$, flip bifurcation point is $r_1 = 2.0433$ (Figure 3 b). For the parameter values $c = 0.25$, $h = 0.4$, $\alpha = 1$, the critical value of Flip bifurcation point is obtained as $r_1 = 6.38532$ (Figure 4 a). For $h = 0.8$, that is $r_1 = 3.19266$ (Figure 4 b).

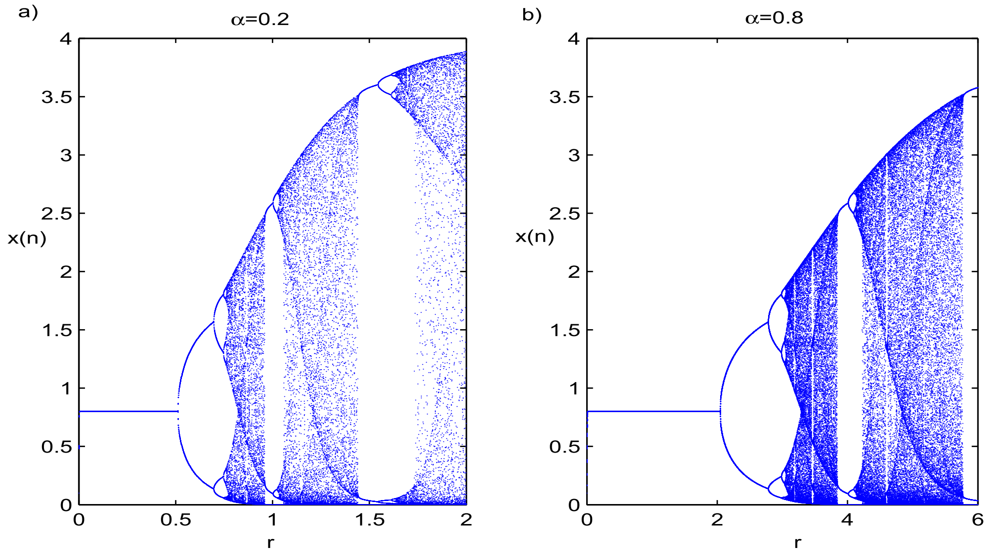

Figure 3. Bifurcation diagram of the system for the parameters $c = 0.25$, $h = 1$.

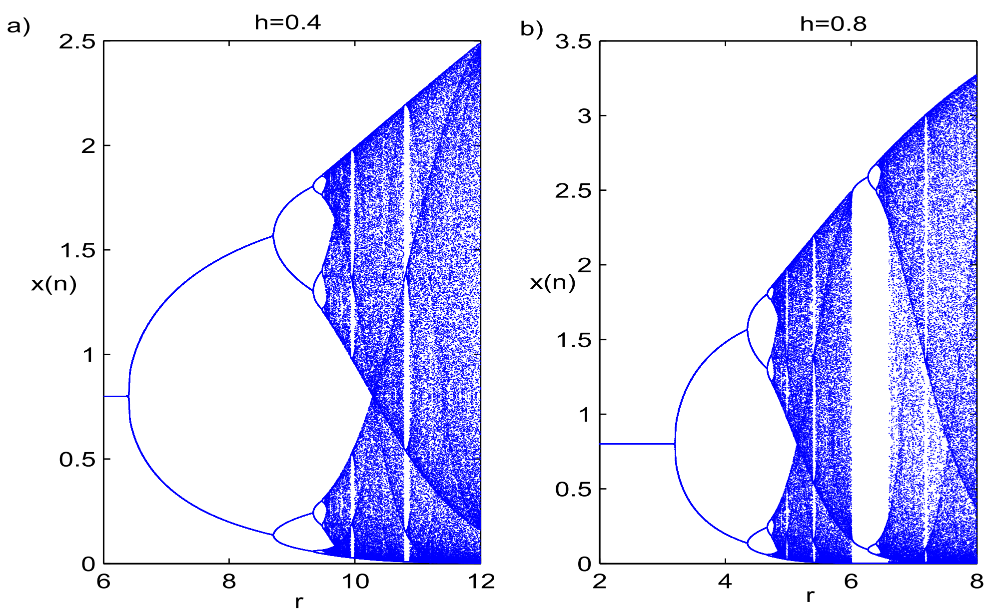

Figure 4. Bifurcation diagram of the system for the parameters $c = 0.25$, $\alpha = 1$.

5. Result and discussion

By using the piecewise constant approximation, a discretization process to discretize conformable fractional logistic differential equation (10) is given in this study. In the interval $t \in [nh, (n+1)h)$, $n = 0, 1, 2, \ldots$, this process gives difference equation (18). Thus the fractional derivative of order α is included as a new parameter into the difference equation. The condition that allows the positive equilibrium point of the equation to be local asymptotically stable is obtained by the inequality (19) according to the parameter r. In addition, the conditions (23) and (24) show the regions where the solutions are eventually monotonic and damped oscillatory and so convergence to the positive equilibrium point respectively (see Figures 1 and 2). We also obtain that the positive equilibrium point of the equation is global asymptotically stable under the condition $c \geq 1$ by using a Lyapunov function.

The occurrence of flip bifurcation in Equation (18) with the bifurcation parameter r is shown theoretically by using the center manifold theory. On the other hand, the effect of the change of fractional derivative order α on Equation (18) is illustrated in Figure 1. This figure shows that the stable behaviour of the equation is destabilizing when decreasing the fractional-order parameter α. For example, while for $\alpha = 0.2$, the flip bifurcation point is $r_1 = 0.510826$; when $\alpha = 0.8$, this value is $r_1 = 2.0433$ (Figures 1 and 3). The discretization parameter h is the other substantial parameter that affects the dynamic structure of the equation. The stable behaviour of Equation (18) is destabilizing when increasing the parameter h (Figure 2). For instance, while for $h = 0.4$, the flip bifurcation point is $r_1 = 6.38532$; when $h = 0.8$, the value of bifurcation point decreases to $r_1 = 3.192664$ (Figure 4).

It is well known that existence or non-existence of chaotic solutions for a dynamical system is determined by calculating Lyapunov exponent. If the system has a positive largest Lyapunov exponent, then the system exhibits chaotic dynamics. In the literature, there are many fractional order systems that exhibit chaotic dynamics [15,21,30–32]. For the model (18), the maximum Lyapunov exponents corresponding to Figures 3 and 4 are calculated and plotted in Figures 5 and 6. These

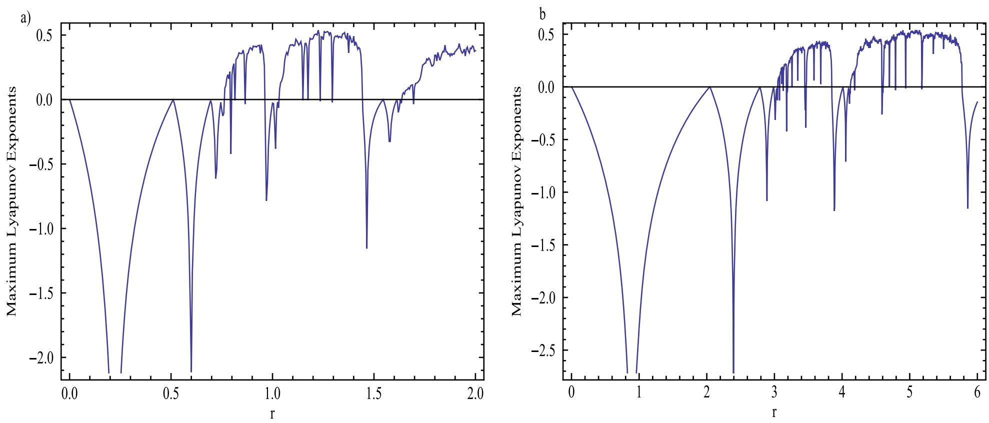

Figure 5. Maximum Lyapunov exponents corresponding to Figure 3(a,b).

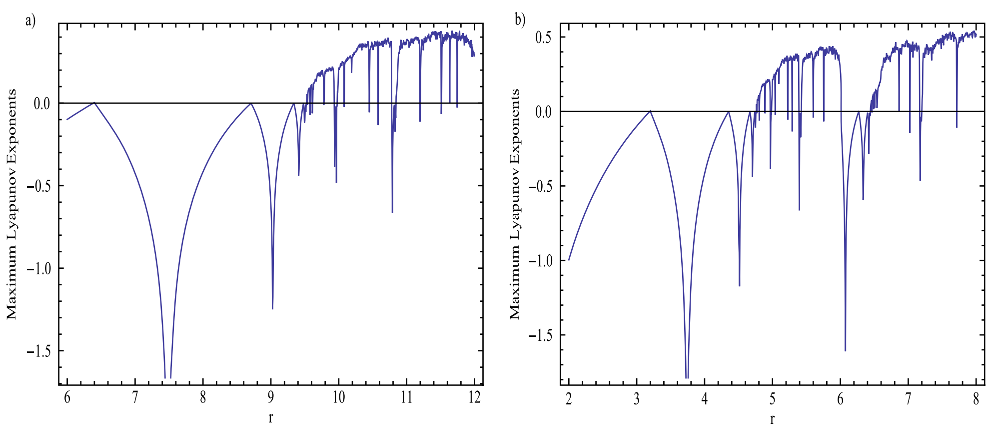

Figure 6. Maximum Lyapunov exponents corresponding to Figure 4(a,b).

figures demonstrate the existence of the chaotic regions and period orbits in the parametric space with increasing the parameter r.

Disclosure statement

No potential conflict of interest was reported by the authors.

References

[1] S. Abbas, M. Banerjee, and S. Momani, *Dynamical analysis of fractional-order modified logistic model*, Comput. Math. Appl. 62 (2011), pp. 1098–1104.

[2] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math. 279 (2015), pp. 57–66.

[3] R.P. Agarwal, A.M.A. El-Sayed, and S.M. Salman, *Fractional-order Chua's system: Discretization, bifurcation and chaos*, Adv. Differ. Equ. 2013 (2013). doi:10.1186/1687-1847-2013-320

[4] I. Ameen and P. Novati, *The solution of fractional order epidemic model by implicit adams methods*, Appl. Math. Model. 43 (2017), pp. 78–84.

[5] A.A.M. Arafa, S.Z. Rida, and M. Khalil, *The effect of anti-viral drug treatment of human immunodeficiency virus type 1 (HIV-1) described by a fractional order model*, Appl. Math. Model. 37 (2013), pp. 2189–2196.

[6] A.A.M. Arafa, S.Z. Rida, and M. Khalil, *A fractional-order model of HIV infection: Numerical solution and comparisons with data of patients*, Int. J. Biomath. 7 (2014), pp. 1450036.

[7] T.M. Atanackovic and B. Stankovic, *An expansion formula for fractional derivatives and its application*, Fract. Calc. Appl. Anal. 7 (2004), pp. 365–378.

[8] S. Bhalekar and V. Daftardar-Gejji, *Fractional ordered Liu system with time-delay*, Commun. Nonlinear Sci. Numer. Simulat. 15 (2010), pp. 2178–2191.

[9] V. Daftardar-Gejji and H. Jafari, *Adomian decomposition: A tool for solving a system of fractional differential equations*, J. Math. Anal. Appl. 301 (2005), pp. 508–518.

[10] Y. Ding and H. Ye, *A fractional-order differential equation model of HIV infection of CD4+ T-cells*, Math. Comput. Model. 50 (2009), pp. 386–392.

[11] V.D. Djordjevic, J. Jaric, B. Fabry, J.J. Fredberg, and D. Stamenovi, *Fractional derivatives embody essential features of cell rheological behavior*, Ann. Biomed. Eng. 31 (2003), pp. 692–699.

[12] M. El-Shahed and A. Salem, *On the generalized Navier–Stokes equations*, Appl. Math. Comput. 156 (2004), pp. 287–293.

[13] Z.F. El-Raheem and S.M. Salman, *On a discretization process of fractional-order Logistic differential equation*, J. Egypt. Math. Soc. 22 (2014), pp. 407–412.

[14] A.M.A. El-Sayed, Z.F. El-Raheem, and S.M. Salman, *Disretization of forced duffing system with fractional-order damping*, Adv. Differ. Equ. 2014 (2014), pp. 66. doi:10.1186/1687-1847-2014-66

[15] A.M.A. El-Sayed, A. Elsonbaty, A.A. Elsadany, and A.E. Matouk, *Dynamical analysis and circuit simulation of a new fractional-order hyperchaotic system and its discretization*, Int. J. Bifurcat. Chaos 26(13) (2016), pp. 1650222.

[16] V.S. Ertürk and S. Momani, *Solving systems of fractional differential equations using differential transform method*, J. Comput. Appl. Math. 215 (2008), pp. 142–151.

[17] V.S. Ertürk, Z.M. Odibat, and S. Momani, *An approximate solution of a fractional order differential equation model of human T-cell Iymphotropic virus I (HTLV-I) infection of CD4+ T cells*, Chaos Soliton Fract. 62 (2011), pp. 996–1002.

[18] K. Gopalsamy and P. Liu, *Persistence and global stability in a population model*, J. Math. Anal. Appl. 224 (1998), pp. 59–80.

[19] M.S. Hashemi, *Invariant subspaces admitted by fractional differential equations with conformable derivatives*, Chaos Soliton Fract. 107 (2018), pp. 161–169.

[20] M.S. Hashemi, *Some new exact solutions of (2+1)-dimensional nonlinear Heisenberg ferromagnetic spin chain with the conformable time fractional derivative*, Opt. Quant. Electron. 107 (2018), pp. 50–79.

[21] A.S. Hegazia, E. Ahmeda, and A.E. Matouk, *On chaos control and synchronization of the commensurate fractional order Liu system*, Commun. Nonlinear Sci. Numer. Simul. 18 (2013), pp. 1193–1202.

[22] H. Jafari and V. Daftardar-Gejji, *Revised Adomian decomposition method for solving systems of ordinary and fractional differential equations*, Appl. Math. Comput. 181 (2006), pp. 598–608.

[23] H. Jafari and V. Daftardar-Gejji, *Solving a system of nonlinear fractional differential equations using Adomian decomposition*, J. Math. Anal. Appl. 196 (2006), pp. 644–651.

[24] D. Jun, Z.G. Jun, X. Yong, Y. Hong, and W. Jue, *Dynamic behavior analysis of fractional-order Hindmarsh–Rose neuronal model*, Cong. Neurodyn. 8 (2014), pp. 167–175.

[25] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math. 264 (2014), pp. 65–70.

[26] N.A. Khan, M. Jamil, A. Ara, and N.U. Khan, *On efficient method for system of fractional differential equations*, Adv. Differ. Equ. 2011 (2011), p. 303472.Article ID 303472.

[27] Y.A. Kuznetsov, *Elements of Applied Bifurcation Theory*, Springer-Verlag, New York, 1998.

[28] J.A.T. Machado, M.F. Silva, R.S. Barbosa, I.S. Jesus, C.M. Reis, M.G. Marcos, and A.F. Galhano, *Some applications of fractional calculus in engineering*, Math. Probl. Eng. 2010 (2010), Article ID 639801.

[29] A.E. Matouk, *Stability conditions, hyperchaos and control in a novel fractional order hyperchaotic system*, Phys. Lett. A 373 (2009), pp. 2166–2173.

[30] A.E. Matouk, *Chaos, feedback control and synchronization of a fractional-order modified Autonomous Van der Pol-Duffing circuit*, Commun. Nonlinear Sci. Numer. Simulat. 16 (2011), pp. 975–986.

[31] A.E. Matouk, *Chaos synchronization of a fractional-order modified Van der Pol-Duffing system via new linear control, backstepping control and Takagi–Sugeno fuzzy approaches*, Complexity 21(S1) (2016), pp. 116–124.

[32] A.E. Matouk and A.A. Elsadany, *Dynamical analysis, stabilization and discretization of a chaotic fractional-order GLV model*, Nonlinear Dyn. 85 (2016), pp. 1597–1612.

[33] A.E. Matouk, A.A. Elsadany, E. Ahmed, and H.N. Agiza, *Dynamical behavior of fractional-order Hastings–Powell food chain model and its discretization*, Commun. Nonlinear Sci. Numer. Simulat. 27 (2015), pp. 153–167.

[34] S. Momani and Z. Odibat, *Numerical approach to differential equations of fractional order*, J. Comput. Appl. Math. 207 (2007), pp. 96–110.

[35] C.A. Monje, Y. Chen, B.M. Vinagre, D. Xue, and V. Feliu-Batlle, *Fractional-order Systems and Controls. Fundamentals and Applications*, Springer, London, 2010.

[36] Z. Odibat and S. Momani, *Numerical methods for nonlinear partial differential equations of fractional order*, Appl. Math. Model. 32 (2008), pp. 28–39.

[37] N. Özal and E. Demirci, *A fractional order SEIR model with vertical transmission*, Math. Comput. Model. 54 (2011), pp. 1–6.

[38] G.G. Parra, A.J. Arenas, and B.M. Chen-Charpentier, *A fractional order epidemic model for the simulation of outbreaks of influenza A(H1N1)*, Math. Method Appl. Sci. 37 (2014), pp. 2218–2226.

[39] L.M. Petrovic, D.T. Spasic, and T.M. Atanackovic, *On a mathematical model of a human root dentin*, Dent. Mater. 21 (2005), pp. 125–128.

[40] C.M.A. Pinto and J.A.T. Machado, *Fractional model for malaria transmission under control strategies*, Comput. Math. Appl. 66 (2013), pp. 908–916.

[41] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[42] F.A. Rihan, *Numerical modelling of fractional-order biological systems*, Abstr. Appl. Anal. 2013 (2013), Article ID 816803.

[43] R. Scherera, S.L. Kallab, Y. Tang, and J. Huang, *The Grünwald–Letnikov method for fractional differential equations*, Comput. Math. Appl. 62 (2011), pp. 902–917.

[44] H.H. Sherief and A.M. Abd El-Latief, *Application of fractional order theory of thermo elasticity to a 2D problem for a half-space*, Appl. Math. Comput. 248 (2014), pp. 584–592.

[45] V.E. Tarasov, *Fractional statistical mechanics*, Chaos 16 (2006), Article ID 033108.

[46] C. Ünlü, H. Jafari, and D. Baleanu, *Revised variational iteration method for solving systems of nonlinear fractional order differential equations*, Abstr. Appl. Anal. 2013 (2013), Article ID 461837.

[47] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, 2nd ed., Springer-Verlag, New York, 2003.

[48] G.C. Wu, *A fractional variational iteration method for solving fractional nonlinear differential equations*, J. Comput. Appl. Math. 61 (2011), pp. 2186–2190.

[49] Y. Wu, C. Wang, and S.J. Liao, *Solving the one-loop soliton solution of the Vakhnenko equation by means of the Homotopy analysis method*, Chaos Soliton Fract. 23 (2005), pp. 1733–1740.

[50] Y. Yan and C. Kou, *Stability analysis for a fractional differential model of HIV infection of CD4+ T-cells with time delay*, Math. Comput. Simulat. 82 (2012), pp. 1572–1585.

[51] H. Ye and Y. Ding, *Nonlinear dynamics and chaos in a fractional-order HIV model*, Math. Probl. Eng. 2009 (2009), Article ID 378614.