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Flip and Neimark–Sacker bifurcation in a differential equation with piecewise constant arguments model

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ABSTRACT

In this paper, a differential equation with piecewise constant arguments model that describes a population density of a bacteria species in a microcosm is considered. The discretization process of a differential equation with piecewise constant arguments gives us two dimensional discrete dynamical system in the interval $t \in [n, n+1)$. By using the center manifold theorem and the bifurcation theory, it is shown that the discrete dynamical system undergoes flip and Neimark–Sacker bifurcation. The bifurcation diagrams, phase portraits and Lyapunov exponents are obtained for the discrete model.

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1. Introduction

The differential equation with piecewise constant arguments includes both discrete and continuous time and so combines properties both differential and difference equation. These equations have attracted great attention from the researchers in mathematics, biology, engineering and other fields [1–19]. Nevertheless, these equations have limited application in biology because the effect of piecewise constant arguments in a population dynamics is not well understood. So it is clear that further studies are needed for the application of differential equation with piecewise constant arguments in a population dynamics. Theoretical studies have shown that differential equation with piecewise constant arguments are equivalent to integral equations and are very close to delay differential equations [1–3]. It is well known that delay differential equations occupy a place of central importance in the population dynamics because the rate of populations may depends on the present size and the memorized values of the population. These biological phenomenon may be described by using differential equations with piecewise constant arguments.

The original method of investigation of these equations was based on the reduction to discrete systems. Using this method, many authors have analyzed various types of differential equation with piecewise constant arguments [2,4–15]. The existence and uniqueness of solutions, oscillations and stability, integral manifolds and periodic solutions, and numerous other issues have been intensively discussed [16–19]. In several papers [2,4–15] authors have investigated different types of population models based on logistic equations with piecewise constant arguments and have obtained mathematical results on oscillations or chaotic behavior. In study [2,4], the simplest logistic equation with piecewise constant

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arguments

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = rx(t)(1 - \frac{x([t])}{K}) \tag{1}$$

has been considered as a semi-discretization of the delay logistic equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = rx(t)(1 - \frac{x(t-1)}{K}) \tag{2}$$

where $[t]$ denotes the integer part of $t \in [0, \infty)$. Gopalsamy and Liu [5], studied a more general equation in the following form:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = rx(t)(1 - ax(t) - bx([t])). \tag{3}$$

They showed that all positive solutions of equation (3) converge to the positive equilibrium points. The other studies about the equation (3) can be found in the studies [8–11].

Following these works, Gurcan and Bozkurt [12] studied the differential equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = rx(t)(1 - \alpha x(t) - \beta_0 x([t]) - \beta_1 x([t-1])) \tag{4}$$

where the parameters r, α, β_0 and β_1 are positive numbers. They obtained some theoretical results for the local and global dynamics of the equation. In addition equation (4) has some application in a population dynamics [13].

In the literature, there are limited number of studies discussing the qualitative behavior of the logistic equation with piecewise constant arguments, which include bifurcations and chaos phenomena [6,7,20]. May [6] obtained that difference equation (1) can be complex and exhibits chaotic dynamics for the parameter values of r. In study [7], the authors showed that for certain parameter values of a and b, equation (3) generates Li-Yorke chaos.

The purpose of this paper is to investigate possible bifurcation type of model (4) such us Flip and Neimark–sacker bifurcation using center manifold and bifurcation theory.

This paper is organized as follows: In Section 2, we first give the local stability conditions of positive equilibrium point of the equation (4). In Section 3, we investigate possible bifurcation type of model (4) and show that the model enters flip bifurcation and Neimark–Sacker bifurcation. By using center manifold theorem and bifurcation theorem, we obtain the direction and stability of the both Flip and Neimark–Sacker bifurcation. Theoretical results are verified by numerical simulations for two examples which included phase portrait, bifurcation diagrams, Lyapunov exponents. Finally, Section 4 draws the conclusion.

2. Local stability analysis

The discretization of the equation (4) in the interval $t \in [n, n+1)$ can be obtained as the following difference Equation [12,13]:

$$x(n+1) = \frac{x(n)(1 - \beta_0 x(n) - \beta_1 x(n-1))}{(1 - \alpha x(n) - \beta_0 x(n) - \beta_1 x(n-1))e^{-r(1-\beta_0 x(n) - \beta_1 x(n-1))} + \alpha x(n)}. \tag{5}$$

If we introduce $u_1(n) = x(n)$ and $u_2(n) = x(n-1)$, Equation (5) can be rewritten as

$$\begin{cases} u_1(n+1) = \frac{u_1(n)(1-\beta_0 u_1(n)-\beta_1 u_2(n))}{(1-\alpha u_1(n)-\beta_0 u_1(n)-\beta_1 u_2(n))e^{-r(1-\beta_0 u_1(n)-\beta_1 u_2(n))}+\alpha u_1(n)}, \\ u_2(n+1) = u_1(n). \end{cases} \tag{6}$$

Now the discrete dynamical system (6) reveals the dynamical characteristics of the system of differential equations with piecewise constant arguments (4). Therefore, we will continue to analyze the system of (6) instead of Equation (4).

The positive equilibrium point of system (6) is

$$(\overline{u_1}, \overline{u_2}) = \left(\frac{1}{\alpha+\beta_0+\beta_1}, \frac{1}{\alpha+\beta_0+\beta_1}\right). \tag{7}$$

Let $u(n+1) = Ju(n)$ is linearized system of (6) about $(\overline{u_1}, \overline{u_2})$. So, the Jacobian matrix J can be calculated as

$$\mathbf{A(r)} = \mathbf{J}(\overline{\mathbf{u_1}}, \overline{\mathbf{u_2}}) = \begin{pmatrix} \left(1+\frac{\beta_0}{\alpha}\right)e^{\frac{-r\alpha}{\alpha+\beta_0+\beta_1}} - \frac{\beta_0}{\alpha} & \frac{\beta_1}{\alpha}\left(e^{\frac{-r\alpha}{\alpha+\beta_0+\beta_1}}-1\right) \\ 1 & 0 \end{pmatrix} \tag{8}$$

which gives the characteristic equation

$$p(\lambda) = \lambda^2 + \lambda\left(-\left(1+\frac{\beta_0}{\alpha}\right)e^{\frac{-r\alpha}{\alpha+\beta_0+\beta_1}} + \frac{\beta_0}{\alpha}\right) - \left(\frac{\beta_1}{\alpha}e^{\frac{-r\alpha}{\alpha+\beta_0+\beta_1}} - \frac{\beta_1}{\alpha}\right) = 0. \tag{9}$$

Using the characteristic Equation (9), the local stability conditions of the system (6) can be obtained as the following theorem.

Theorem 2.1 [12,13]: *Let $\beta_0 > \alpha + \beta_1 > 2\alpha$. The following statements are true.*

(a) *Assume that $3\beta_1 < \alpha + \beta_0$. The positive equilibrium point of system (6) is local asymptotically stable if and only if*

$$0 < r < \frac{\alpha+\beta_0+\beta_1}{\alpha}\ln\left(\frac{\alpha+\beta_0-\beta_1}{\beta_0-\alpha-\beta_1}\right). \tag{10}$$

(b) *Assume that $3\beta_1 > \alpha + \beta_0$. The positive equilibrium point of system (6) is local asymptotically stable if and only if*

$$0 < r < \frac{\alpha+\beta_0+\beta_1}{\alpha}\ln\left(\frac{\beta_1}{\beta_1-\alpha}\right). \tag{11}$$

Example 2.2: The parameter values $r = 2.3$, $\alpha = 0.4$, $\beta_0 = 1.2$ and $\beta_1 = 0.1$ satisfy the condition of Theorem (2.1)a. Using these parameters values and the initial conditions $u_1(1) = 0.4$, $u_2(1) = 0.45$, we hold Figure 1 which shows that the positive equilibrium point $(\overline{u_1}, \overline{u_2}) = (0.588235, 0.588235)$ is local asymptotically stable.

3. Bifurcation analysis

In this section, we first investigate the existence of possible bifurcation type for the system (6). Stationary bifurcation does not exist for the system (6) because we always hold $p(1) =$

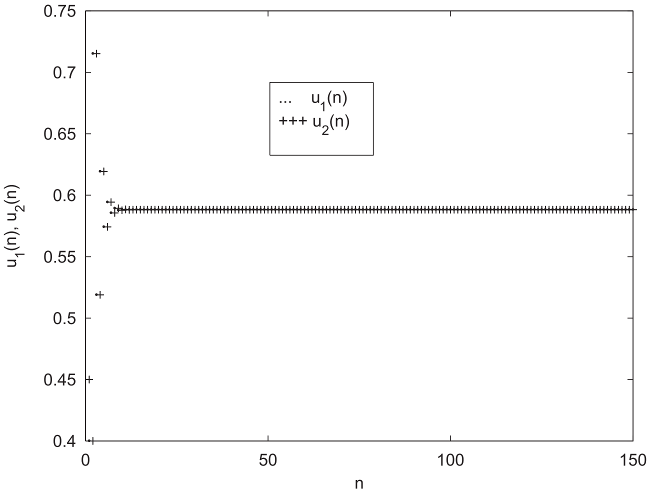

Figure 1. A stable equilibrium point for the system (6).

$(1+\frac{\beta_0}{\alpha}+\frac{\beta_1}{\alpha})(1-e^{\frac{-r\alpha}{\alpha+\beta_0+\beta_1}})\neq 0$ [21]. The other bifurcations such as Flip and Neimark–Sacker bifurcation are studied in the following section.

3.1. Flip bifurcation

To study Flip bifurcation, the parameter r is chosen as a bifurcation parameter. By using the bifurcation theory in [21,23–35], we will investigate the conditions and direction of Flip bifurcation.

Theorem 3.1 [21,22]: *For the system (6), one of the eigenvalues is -1 and the other eigenvalue lie inside the unit circle if and only if*

$$\begin{aligned}
&\text{(a) } p(1)=1+p_1+p_0>0,\\
&\text{(b) } p(-1)=1-p_1+p_0=0,\\
&\text{(c) } D_1^{+}=1+p_0>0,\\
&\text{(d) } D_1^{-}=1-p_0>0.
\end{aligned}$$

Lemma 3.2 (Eigenvalue Assignment): *Let $\beta_0>\alpha+\beta_1>2\alpha$ and $3\beta_1<\alpha+\beta_0$.If*

$$r_1=\frac{\alpha+\beta_0+\beta_1}{\alpha}\ln\left(\frac{\alpha+\beta_0-\beta_1}{\beta_0-\alpha-\beta_1}\right),$$

then the eigenvalue assignment condition of Flip bifurcation in Theorem (3.1) holds.

Proof: By considering the characteristic Equation (9), we obtain

$$p_1=-\left(1+\frac{\beta_0}{\alpha}\right)e^{\frac{-r\alpha}{\alpha+\beta_0+\beta_1}}+\frac{\beta_0}{\alpha}, \tag{12}$$

$$p_0=\frac{\beta_1}{\alpha}\left(1-e^{\frac{-r\alpha}{\alpha+\beta_0+\beta_1}}\right). \tag{13}$$

The condition (a) Theorem (3.1) gives the inequality

$$p(1) = \left(1 + \frac{\beta_0}{\alpha} + \frac{\beta_1}{\alpha}\right)\left(1 - e^{\frac{-r\alpha}{\alpha+\beta_0+\beta_1}}\right) > 0 \tag{14}$$

which always hold. Considering the condition (b) with the fact $\beta_0 > \alpha + \beta_1$, we have

$$p(-1) = \frac{\alpha - \beta_0 + \beta_1}{\alpha} + \left(\frac{\alpha + \beta_0 - \beta_1}{\alpha}\right) e^{\frac{-r\alpha}{\alpha+\beta_0+\beta_1}} = 0 \tag{15}$$

which gives

$$r_1 = \frac{\alpha + \beta_0 + \beta_1}{\alpha} \ln\left(\frac{\alpha + \beta_0 - \beta_1}{\beta_0 - \alpha - \beta_1}\right).$$

From (c), we get that the inequality

$$D_1^+ = 1 + \frac{\beta_1}{\alpha}(1 - e^{\frac{-r\alpha}{\alpha+\beta_0+\beta_1}}) > 0 \tag{16}$$

is always satisfied. Computing the condition (d) with the fact $\beta_1 > \alpha$, we have

$$D_1^- = \frac{\alpha - \beta_1}{\alpha} + \frac{\beta_1}{\alpha} e^{\frac{-r\alpha}{\alpha+\beta_0+\beta_1}} > 0 \tag{17}$$

which leads to

$$0 < r < \frac{\alpha + \beta_0 + \beta_1}{\alpha} \ln\left(\frac{\beta_1}{\beta_1 - \alpha}\right).$$

Under the condition $3\beta_1 < \alpha + \beta_0$, we have

$$r_1 = \frac{\alpha + \beta_0 + \beta_1}{\alpha} \ln\left(\frac{\alpha + \beta_0 - \beta_1}{\beta_0 - \alpha - \beta_1}\right) < \frac{\alpha + \beta_0 + \beta_1}{\alpha} \ln\left(\frac{\beta_1}{\beta_1 - \alpha}\right). \tag{18}$$

This completes the proof. □

Now, it is easy to check that the Jacobian matrix J has the eigenvalues

$$\lambda_1(r_1) = -1 \quad \text{and} \quad \lambda_2(r_1) = \frac{2\beta_1}{-\alpha - \beta_0 + \beta_1}$$

which shows the correctness Lemma (3.2). We note that the condition $3\beta_1 < \alpha + \beta_0$ given in Lemma (3.2) leads to $|\lambda_2(r_1)| \neq 1$ and under the conditions of Lemma (3.2), it holds that $|\lambda_2(r_1)| < 1$.

To compute the coefficients of normal form, we convert the origin of the coordinates to equilibrium point $(\overline{u_1}, \overline{u_2}) = (\frac{1}{\alpha+\beta_0+\beta_1}, \frac{1}{\alpha+\beta_0+\beta_1})$ by the change of variables

$$\begin{cases} u_1 = \overline{u_1} + x_1, \\ u_2 = \overline{u_2} + x_2, \end{cases} \tag{19}$$

This transforms system (6) into

$$\begin{cases} x_1(n+1) = \dfrac{\left(x_1(n)+\frac{1}{\alpha+\beta_0+\beta_1}\right)\left(-\beta_0 x_1(n)-\beta_1 x_2(n)+\frac{\alpha}{\alpha+\beta_0+\beta_1}\right)}{-e^{r\left(\beta_0 x_1(n)+\beta_1 x_2(n)-\frac{\alpha}{\alpha+\beta_0+\beta_1}\right)}\left(x_1(n)(\alpha+\beta_0)+\beta_1 x_2(n))+\alpha(x_1(n)+\frac{1}{\alpha+\beta_0+\beta_1}\right)}, \\ x_2(n+1) = x_1(n) + \frac{1}{\alpha+\beta_0+\beta_1}, \end{cases} \tag{20}$$

This system can be rewritten in the form

$$X_{n+1} = F_i(X_n, r), i = 1, 2. \tag{21}$$

For map (20), we have

$$X_{n+1} = JX_n + \frac{1}{2}B(X_n, X_n) + \frac{1}{6}C(X_n, X_n, X_n) + O(X_n^4), \tag{22}$$

where

$$J = \mathbf{A}(r_1) = \begin{pmatrix} -\frac{\alpha+\beta_0+\beta_1}{\alpha+\beta_0-\beta_1} & -\frac{2\beta_1}{\alpha+\beta_0-\beta_1} \\ 1 & 0 \end{pmatrix} \tag{23}$$

and the multilinear functions B and C are defined by

$$B_i(x, y) = \sum_{j,k=1}^{2} \frac{\partial^2 F_i(\varepsilon, 0)}{\partial \varepsilon_j \partial \varepsilon_k}|_{\varepsilon=0} x_j y_k, i = 1, 2$$

and

$$C_i(x, y, z) = \sum_{j,k,l=1}^{2} \frac{\partial^3 F_i(\varepsilon, 0)}{\partial \varepsilon_j \partial \varepsilon_k \partial \varepsilon_l}|_{\varepsilon=0} x_j y_k z_l, i = 1, 2.$$

For the system (20), the values of B and C can be obtained as

$$B(\varepsilon, \eta) = \begin{pmatrix} \delta_1\varepsilon_1\eta_1 + \delta_2\varepsilon_1\eta_2 + \delta_3\varepsilon_2\eta_1 + \delta_4\varepsilon_2\eta_2 \\ 0 \end{pmatrix}, \tag{24}$$

and

$$C(\varepsilon, \eta, \zeta) = \begin{pmatrix} \varepsilon_1\eta_1(\varphi_1\zeta_1 + \varphi_2\zeta_2) + \varepsilon_1\eta_2(\varphi_3\zeta_1 + \varphi_4\zeta_2) + \varepsilon_2\eta_1(\varphi_5\zeta_1 + \varphi_6\zeta_2) + \varepsilon_2\eta_2(\varphi_7\zeta_1 + \varphi_8\zeta_2) \\ 0 \end{pmatrix}. \tag{25}$$

where

$$\begin{cases}\delta_1 = & \frac{2e^{\frac{-2r\alpha}{\alpha+\beta_0+\beta_1}}(\alpha+\beta_0)}{\alpha^2}((\alpha+\beta_0)(\alpha+\beta_0+\beta_1) \\ & -e^{\frac{r\alpha}{\alpha+\beta_0+\beta_1}}(\beta_0^2+\alpha(\alpha+\beta_1)+\beta_0(2\alpha-r\alpha+\beta_1))), \\ \delta_2 = & \delta_3 = \\ & \frac{e^{\frac{-2r\alpha}{\alpha+\beta_0+\beta_1}}\beta_1}{\alpha^2}(2(\alpha+\beta_0)(\alpha+\beta_0+\beta_1)+e^{\frac{r\alpha}{\alpha+\beta_0+\beta_1}} \\ & ((-2+r)\alpha^2+2\beta_0(-2\alpha+r\alpha-\beta_0)-2\beta_1(\alpha+\beta_0))), \\ \delta_4 = & \frac{2e^{\frac{-2r\alpha}{\alpha+\beta_0+\beta_1}}\beta_1^2}{\alpha^2}(\alpha+\beta_0+\beta_1-e^{\frac{r\alpha}{\alpha+\beta_0+\beta_1}}(\alpha-r\alpha+\beta_0+\beta_1)).\end{cases} \tag{26}$$

$$\begin{cases}\varphi_1 = \frac{3}{\alpha^3}e^{\frac{-3r\alpha}{\alpha+\beta_0+\beta_1}}(\alpha+\beta_0)(2(\alpha+\beta_0)^2(\alpha+\beta_0+\beta_1)^2 \\ \quad -2e^{\frac{r\alpha}{\alpha+\beta_0+\beta_1}}(\alpha+\beta_0)(\alpha+\beta_0+\beta_1)(\beta_0^2+2\alpha(\alpha+\beta_1)+\beta_0((3-2r)\alpha+\beta_1)) \\ \quad +e^{\frac{2r\alpha}{\alpha+\beta_0+\beta_1}}\alpha(-2(-1+r)\beta_0^3-2\beta_0((-3+r)\alpha-\beta_1)(\alpha+\beta_1)+2\alpha(\alpha+\beta_1)^2 \\ \quad +\beta_0^2((6+(-4+r)r)\alpha-2(-2+r)\beta_1))), \\ \varphi_2 = \varphi_3 = \varphi_5 \\ \quad = \frac{\beta_1}{\alpha^3}e^{\frac{-3r\alpha}{\alpha+\beta_0+\beta_1}}(2e^{\frac{r\alpha}{\alpha+\beta_0+\beta_1}}(\alpha+\beta_0)(-3\beta_0^2+\alpha((-5+2r)\alpha-5\beta_1) \\ \quad +\beta_0(-8\alpha+6r\alpha-3\beta_1))(\alpha+\beta_0+\beta_1)+6(\alpha+\beta_0)^2(\alpha+\beta_0+\beta_1)^2 \\ \quad +e^{\frac{2r\alpha}{\alpha+\beta_0+\beta_1}}\alpha(-2(-2+r)\alpha^3 \\ \quad +\beta_0(2(-3+r)(-2+r)\alpha^2+(12+r(-14+3r))\alpha\beta_0+(4-6r)\beta_0^2) \\ \quad -2(\alpha+\beta_0)((-4+r)\alpha+(-4+3r)\beta_0)\beta_1+4(\alpha+\beta_0)\beta_1^2)), \\ \varphi_4 = \varphi_6 = \varphi_7 \\ \quad = \frac{\beta_1^2}{\alpha^3}e^{\frac{-3r\alpha}{\alpha+\beta_0+\beta_1}}(6(\alpha+\beta_0)(\alpha+\beta_0+\beta_1)^2 \\ \quad -2e^{\frac{r\alpha}{\alpha+\beta_0+\beta_1}}(\alpha+\beta_0+\beta_1)(3\beta_0^2+4\alpha(\alpha-r\alpha+\beta_1)+\beta_0((7-6r)\alpha+3\beta_1)) \\ \quad +e^{\frac{2r\alpha}{\alpha+\beta_0+\beta_1}}\alpha((2+(-4+r)r)\alpha^2+(2-6r)\beta_0^2+2\beta_1(-2(-1+r)\alpha+\beta_1) \\ \quad +\beta_0((4+r(-10+3r))\alpha+(4-6r)\beta_1))), \\ \varphi_8 = \frac{3\beta_1^3}{\alpha^3}e^{\frac{-3r\alpha}{\alpha+\beta_0+\beta_1}}(e^{\frac{2r\alpha}{\alpha+\beta_0+\beta_1}}r\alpha((-2+r)\alpha-2\beta_0-2\beta_1) \\ \quad +2e^{\frac{r\alpha}{\alpha+\beta_0+\beta_1}}((-1+2r)\alpha-\beta_0-\beta_1)(\alpha+\beta_0+\beta_1)+2(\alpha+\beta_0+\beta_1)^2).\end{cases} \tag{27}$$

It is well know that A has simple eigenvalue $\lambda_1(r_1) = -1$, and the corresponding eigenspace E^c is one dimensional and spanned by an eigenvector $q \in R^2$ such that $A(r_1)q = -q$. Let $p \in R^2$ be the adjoint eigenvector, that is, $A^T(r_1)p = -p$. By direct calculation we obtain

$$q \sim (-1,1)^T,$$
$$p \sim \left(\frac{\alpha+\beta_0-\beta_1}{2\beta_1}, 1\right)^T.$$

To obtain the necessary normalization $\langle p, q\rangle = 1$, we can choose

$$q = (-1, 1)^T,$$
$$p = \left(-\frac{\alpha+\beta_0-\beta_1}{\alpha+\beta_0-3\beta_1}, -\frac{2\beta_1}{\alpha+\beta_0-3\beta_1}\right)^T.$$

In order to determine the direction of the flip bifurcation, we compute the critical normal form coefficient $c(0)$ by using the following formula:

$$c(0) = \frac{1}{6}\langle p, C(q, q, q)\rangle - \frac{1}{2}\langle p, B(q, (A-I)^{-1}B(q, q))\rangle. \tag{28}$$

From the above analysis and Section 5.4 in [25], Section 3 in [31,32], we have following theorem.

Theorem 3.3: *Suppose that* $(\overline{u_1}, \overline{u_2})$ *is the positive equilibrium point of the system* (6). *If the Lemma* (3.2) *holds and* $c(0) \neq 0$, *then system* (6) *undergoes a flip bifurcation at the equilibrium point* $(\overline{u_1}, \overline{u_2})$ *when the parameter* r *varies in a small neighborhood of* r_1. *Moreover if* $c(0) > 0$ *(respectively,* $c(0) < 0$*), then the period-2 orbits that bifurcate from* $(\overline{u_1}, \overline{u_2})$ *are stable (respectively, unstable).*

Now, we present the bifurcation diagrams, phase portraits and maximum Lyapunov exponents for the system to confirm the above theoretical analysis and show the complex dynamical behaviors by using numerical simulations.

Example 3.4: For the parameters values $\alpha = 0.4$, $\beta_0 = 1.2$ and $\beta_1 = 0.1$, the critical value of Flip bifurcation point is obtained as $r_1 = 3.2391$. Now, the Jacobian matrix corresponding to the system (20) is

$$J = \mathbf{A}(r_1) = \begin{pmatrix} -1.13333 & -0.13333 \\ 1 & 0 \end{pmatrix} \tag{29}$$

Using the formulas (26) and (27), the values of δ_i and φ_i in the multilinear functions B and C can be obtained as

$$\begin{cases} \delta_1 = 0.971598, \\ \delta_2 = \delta_3 = 0.211883, \\ \delta_4 = 0.02269. \end{cases}$$

and

$$\begin{cases} \varphi_1 = 5.02537, \\ \varphi_2 = \varphi_3 = \varphi_5 = -0.0864671, \\ \varphi_4 = \varphi_6 = \varphi_7 = -0.0181984, \\ \varphi_8 = -0.00117201. \end{cases}$$

Now the eigenvectors $q, p \in R^2$ corresponding to $\lambda_1(r_1) = -1$ are

$$q \sim (-0.707107, 0.707107)^T$$

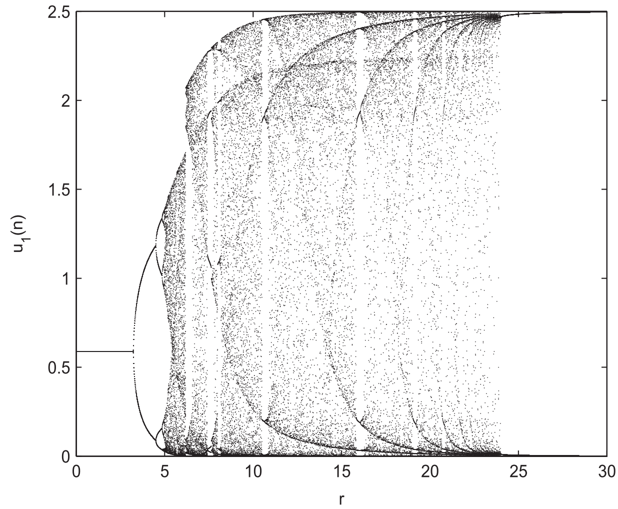

Figure 2. Bifurcation diagram of system (6) in (r, u_1) plane for $\alpha = 0.4$, $\beta_0 = 1.2$, $\beta_1 = 0.1$.

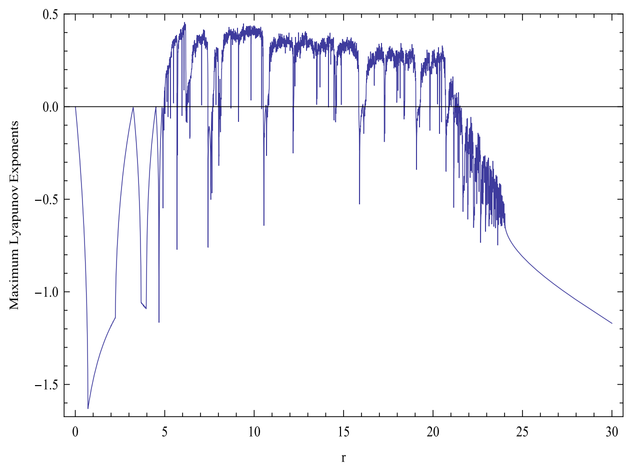

Figure 3. Maximum Lyapunov exponents corresponding to Figure 2.

and

$$p \sim (-0.991228, -0.13216)^T.$$

To achieve the necessary normalization $\langle p, q\rangle = 1$, we can obtain

$$q = (-0.707107, 0.707107)^T,$$
$$p = (-1.63178, -0.217565)^T.$$

Finally, using the formula (28), the critical normal form coefficient $c(0)$ is computed as $c(0) = 0.736539$. Therefore, a unique and stable period-two cycle bifurcates from $(\overline{u_1}, \overline{u_2})$ for $r > r_1 = 3.2391$.

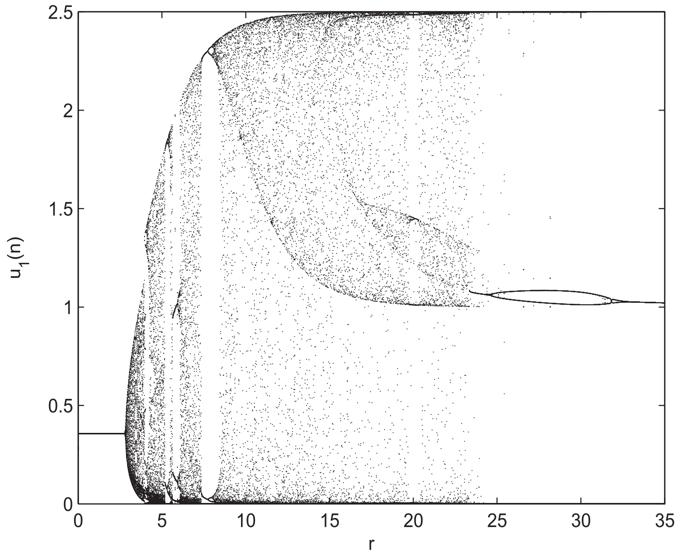

Figure 4. Bifurcation diagram of system (6) in (r, u_1) plane for $\alpha = 0.4$, $\beta_0 = 1.2$, $\beta_1 = 1.2$.

From Figure 2, we observe that the positive equilibrium point $(\overline{u_1}, \overline{u_2})$ of the system (6) is stable for $r < 3.2391$ which shows the correctness of our theoretical results. The Flip bifurcation occurs from the fixed point $(0.588235, 0.588235)$ at $r_1 = 3.2391$. In addition, at $r = r_1$, we have $c(0) = 0.736539$, which determines the direction of the Flip bifurcation. It is well known that existence or non-existence of chaotic solutions for a dynamical system is determined by calculating Lyapunov exponent. If the system has a positive largest Lyapunov exponent, then the system exhibits chaotic dynamics. For the system (6), the maximum Lyapunov exponents corresponding Figure 2 are calculated and plotted in Figure 3 [36]. This figure demonstrates the existence of the chaotic regions and period orbits in the parametric space . From Figure 3, it is observed that some Lyapunov exponents are bigger than 0, some are smaller than 0, so there exist stable fixed points or stable period windows in the chaotic region.

Now, we discuss the Neimark–Sacker bifurcation for the model (6) in the following section.

3.2. Neimark–Sacker bifurcation

Theorem 3.5 [21]: *A pair of complex conjugate roots of* (6) *lie on the unit circle if and only if*

$$
\begin{aligned}
&\text{(a) } p(1) = 1 + p_1 + p_0 > 0,\\
&\text{(b) } p(-1) = 1 - p_1 + p_0 > 0,\\
&\text{(c) } D_1^{+} = 1 + p_0 > 0,\\
&\text{(d) } D_1^{-} = 1 - p_0 = 0.
\end{aligned}
$$

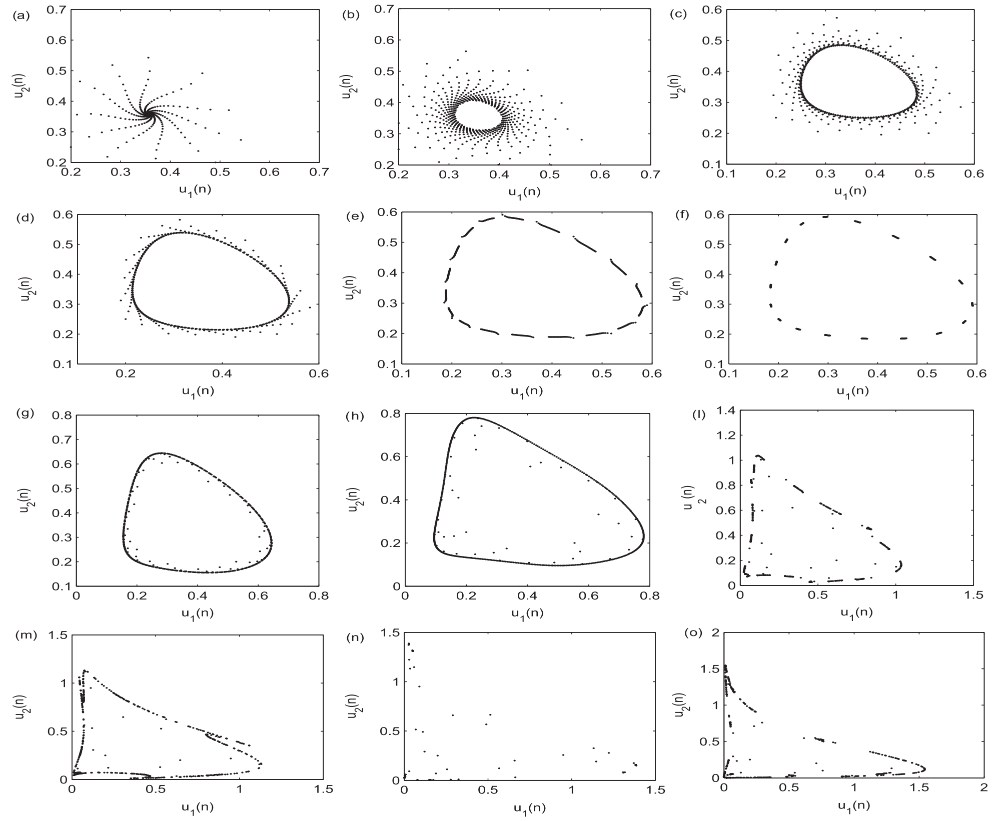

Figure 5. Phase portraits for values of r for the parameters values $\alpha = 0.4$, $\beta_0 = 1.2$, $\beta_1 = 1.2$ where $r = 2.75$ (a), $r = 2.83826$ (b), $r = 2.88$ (c), $r = 2.92$ (d), $r = 2.96$ (e), $r = 2.97$ (f), $r = 3.03$ (g), $r = 3.23$ (h), $r = 3.73$ (l), $r = 3.93$ (m), $r = 4.03$ (n), 4.33 (o).

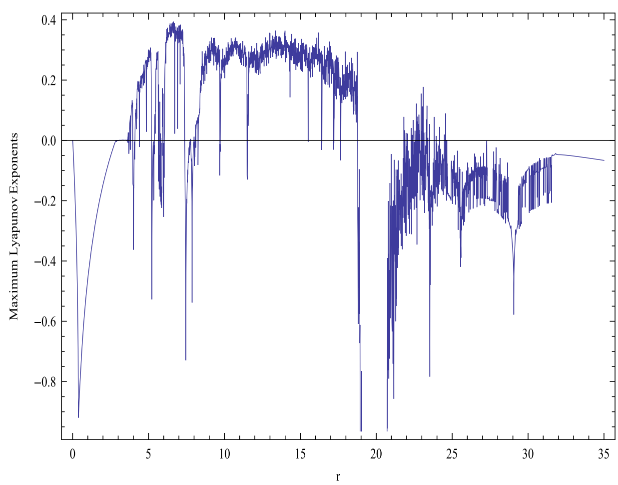

Figure 6. Maximum Lyapunov exponents corresponding to Figure 4.

Lemma 3.6 (Eigenvalue Assignment): *Let $\beta_0 > \alpha + \beta_1 > 2\alpha$ and $3\beta_1 > \alpha + \beta_0$. If*

$$r_2 = \frac{\alpha + \beta_0 + \beta_1}{\alpha} \ln\left(\frac{\beta_1}{\beta_1 - \alpha}\right),$$

then the eigenvalue assignment condition of Neimark–Sacker bifurcation in Theorem (3.5) holds.

Proof: The proof is similar as in Lemma (3.2) and will be omitted. □

It is easy to see that the Jacobian matrix J has the eigenvalues

$$\lambda_{1,2}(r) = \frac{e^{-\frac{r\alpha}{\alpha+\beta_0+\beta_1}}}{2\alpha}(\alpha + \beta_0 - e^{\frac{r\alpha}{\alpha+\beta_0+\beta_1}}\beta_0)$$
$$\pm i\frac{e^{-\frac{r\alpha}{\alpha+\beta_0+\beta_1}}}{2\alpha}\sqrt{4e^{\frac{r\alpha}{\alpha+\beta_0+\beta_1}}\alpha(-\beta_1 + e^{\frac{r\alpha}{\alpha+\beta_0+\beta_1}}\beta_1) - (-\alpha - \beta_0 + e^{\frac{r\alpha}{\alpha+\beta_0+\beta_1}}\beta_0)^2}$$

and for $r = r_2$, these eigenvalues become

$$|\lambda_{1,2}(r_2)| = |\frac{-\alpha - \beta_0 + \beta_1}{2\beta_1} \pm i\frac{\sqrt{4\beta_1^2 - (\alpha + \beta_0 - \beta_1)^2}}{2\beta_1}| = 1.$$

Under the condition $\beta_1 > \alpha$ given in Lemma (3.6), we have

$$\frac{d|\lambda_i(r)|}{dr}|_{r=r_2} = \frac{-\alpha + \beta_1}{2(\alpha + \beta_0 + \beta_1)} \neq 0, \quad i = 1, 2$$

In addition if $trJ(r_2) = -p_1 \neq 0, -1$, which leads to

$$r_2 \neq \frac{\alpha + \beta_0 + \beta_1}{\alpha} \ln\left(\frac{\alpha + \beta_0}{\beta_0}\right), r_2 \neq \frac{\alpha + \beta_0 + \beta_1}{\alpha} \ln\left(\frac{\alpha + \beta_0}{\beta_0 - \alpha}\right),$$

then we have

$$\lambda_i^k(r_2) \neq 1 \quad \text{for} \quad k = 1, 2, 3, 4.$$

Let $q \in C^2$ be an eigenvector of $A(r_2)$ corresponding to the eigenvalue $\lambda_1(r_2)$ such that $A(r_2)q = e^{i\theta_0}q$, and let $p \in C^2$ be an eigenvector of the transposed matrix $A^T(r_2)$ corresponding to its eigenvalue $\overline{\lambda_1(r_2)}$ such that $A^T(r_2)p = e^{-i\theta_0}p$. By direct calculation, we have

$$q \sim \left(\frac{-\alpha - \beta_0 + \beta_1}{2\beta_1} + i\frac{\sqrt{4\beta_1^2 - (\alpha + \beta_0 - \beta_1)^2}}{2\beta_1}, 1\right)^T$$

and

$$p \sim \left(\frac{\alpha + \beta_0 - \beta_1}{2\beta_1} + i\frac{\sqrt{4\beta_1^2 - (\alpha + \beta_0 - \beta_1)^2}}{2\beta_1}, 1\right)^T.$$

To obtain the normalization $\langle p, q\rangle = 1$, we can take

$$q = \left(\frac{-\alpha - \beta_0 + \beta_1}{2\beta_1} + i\frac{\sqrt{4\beta_1^2 - (\alpha + \beta_0 - \beta_1)^2}}{2\beta_1}, 1 \right)^T$$

and

$$p = \left(i\frac{\beta_1}{\sqrt{-(\alpha + \beta_0 - 3\beta_1)(\alpha + \beta_0 + \beta_1)}}, \frac{1}{2} + i\frac{\alpha + \beta_0 - \beta_1}{2\sqrt{-(\alpha + \beta_0 - 3\beta_1)(\alpha + \beta_0 + \beta_1)}} \right)^T.$$

Now we form

$$x = zq + \overline{zq}.$$

In this way, system (20) can be transformed for sufficiently small $|r|$ into following form:

$$z \mapsto \lambda_1(r)z + g(z, \overline{z}, r),$$

where $\lambda_1(r)$ can be written as $\lambda_1(r) = (1 + \varphi(r))e^{i\theta(r)}$ (where $\varphi(r)$ is a smooth function with $\varphi(r_2) = 0$) and g is a complex-valued smooth function. The Taylor expression of g with respect to $(z, \overline{z}) = (0, 0)$ is

$$g(z, \overline{z}, r) = \sum_{k+l\geq 2} \frac{1}{k!l!} g_{kl}(r) z^k \overline{z}^{-l},$$

where

$$\begin{cases} g_{20}(r_2) = \langle p, B(q, q)\rangle, \\ g_{11}(r_2) = \langle p, B(q, \overline{q})\rangle, \\ g_{21}(r_2) = \langle p, C(q, q, \overline{q})\rangle, \\ g_{02}(r_2) = \langle p, B(\overline{q}, \overline{q})\rangle. \end{cases} \tag{30}$$

Now, the coefficient $a(0)$, which determines the direction of the appearance of the invariant curve in a generic system exhibiting Neimark–Sacker bifurcation, can be computed via

$$a(0) = Re\left[\frac{e^{-i\theta_0} g_{21}}{2}\right] - Re\left[\frac{(1 - 2e^{i\theta_0})e^{-2i\theta_0}}{2(1 - e^{i\theta_0})} g_{20} g_{11}\right] - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2. \tag{31}$$

For the above argument and Section 4.7 in [25], we have the following theorem.

Theorem 3.7: *Suppose that $(\overline{u_1}, \overline{u_2})$ is the positive equilibrium point. If the Lemma (3.6) holds, $r_2 \neq \frac{\alpha+\beta_0+\beta_1}{\alpha} \ln\left(\frac{\alpha+\beta_0}{\beta_0}\right)$, $r_2 \neq \frac{\alpha+\beta_0+\beta_1}{\alpha} \ln\left(\frac{\alpha+\beta_0}{\beta_0-\alpha}\right)$ and $a(0) < 0$ (respectively $a(0) > 0$), then the Neimark–Sacker bifurcation of system (6) at $r = r_2$ is supercritical (respectively, subcritical) and there exists a unique closed invariant curve bifurcation from $(\overline{u_1}, \overline{u_2})$ for $r = r_2$, which is asymptotically stable (respectively, unstable).*

Example 3.8: For the parameters values $\alpha = 0.4, \beta_0 = 1.2, \beta_1 = 1.2$, we have critical Neimark–Sacker bifurcation point as $r_2 = 2.83826$. In this situation, the eigenvalues are

$$| \lambda_{1,2}(r) | = | -0.166667 \pm 0.986013i | .$$

In addition it is easy to check that

$$\frac{d|\lambda_i(r)|}{\mathrm{d}r}|_{r=r_2} = 0.142857 \neq 0 \quad \text{and} \quad \lambda_i^k(r_2) \neq 1 \quad \text{for} \quad k = 1, 2, 3, 4.$$

For $r_2 = 2.83826$, the Jacobian matrix J at the fixed point is

$$J = \mathbf{A}(\overline{r_2}) = \begin{pmatrix} -0.333333 & -1 \\ 1 & 0 \end{pmatrix} \tag{32}$$

and has the eigenvalues

$$\lambda_{1,2}(r_2) = -0.166667 \pm 0.986013i = e^{\pm i\theta_0}, \theta_0 = 1.73824.$$

Let $q, p \in C^2$ be complex eigenvectors corresponding to $\lambda_{1,2}$ respectively.

$$q \sim (0.707107, -0.117851 - 0.697217i)^T$$

and

$$p \sim (0.707107, 0.117851 - 0.697217i)^T$$

satisfy

$$A(r_2)q = e^{1.73824i}q,$$
$$A^T(r_2)p = e^{-1.73824i}p.$$

To obtain the normalization $\langle p, q\rangle = 1$, we can take the normalized vectors as

$$q = (0.707107, -0.117851 - 0.697217i)^T$$

and

$$p = (0.707107 - 0.119523i, 1.249\text{x}10^{-16} - 0.717137i)^T.$$

By using the formula (30) the coefficients of the normal of the system (20) can be computed as follows.

$$g_{20}(r_2) = -1.47428 - 0.64862i$$
$$g_{11}(r_2) = 0.12624 + 0.0213385i$$
$$g_{21}(r_2) = 3.48349 + 0.492895i$$
$$g_{02}(r_2) = -1.60556 + 0.128031i.$$

From (31), the critical real part is obtained as $a(0) = -0.86466$. Therefore, a supercritical Neimark–Sacker bifurcation occurs at $r_2 = 2.83826$ (Figure 4).

The bifurcations diagrams of system (6) in the $(r - u_1)$ is given in Figure 4. Numerical studies show that the Neimark–Sacker bifurcation occurs from the equilibrium point $(\overline{u_1}, \overline{u_2}) = (0.357143, 0.357143)$ at $r_2 = 2.83826$. For $r_2 = 2.83826$, we have $|\lambda_{1,2}| = |-0.166667 \pm 0.986013i| = 1$ and $a(0) = -0.86466$ which show that the Neimark–Sacker bifurcation is supercritical. The phase portrait of the system for increasing value of

r is obtained in Figure 5. This figure demonstrates the process of how a smooth invariant circle appears and then disappears from the fixed point. When r exceeds 2.83826, there appears a circular curve enclosing the fixed points. In addition the maximum Lyapunov exponents corresponding to Figure 4 are given in Figure 6.

4. Conclusion

The present study deals with the dynamics of a discrete model, which is based on the discretization of a differential equation with piecewise constant arguments model. The discrete model (6) exhibits the dynamic behavior of the system of differential equations with piecewise constant arguments (4). Therefore, we will continue to analyze the system of (6) instead of equation (4). The stability of fixed point and bifurcations of discrete dynamical system are investigated. The Flip bifurcation and Neimark–Sacker bifurcation of this discrete dynamical system are studied by using center manifold theorem and bifurcation theory. We choose the parameter r as a Flip bifurcation and Neimark–Sacker bifurcation parameter and show that bifurcation happens at certain bifurcation parameter r and under some conditions on parameters α, β_0 and β_1. The Lyapunov exponents are numerically computed to confirm further the complexity of the dynamical behaviors.

Disclosure statement

No potential conflict of interest was reported by the author.

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