***ORIGINAL PAPER***

# MODIFIED GENERALIZED FIBONACCI AND LUCAS QUATERNIONS

SURE KOME[1], CAHIT KOME[1], YASIN YAZLIK[1]

*Manuscript received: 07.09.2018; Accepted paper: 18.01.2019;*
*Published online: 30.03.2019.*

***Abstract.*** *In this paper, we introduce the modified generalized Fibonacci and Lucas quaternions. We present the generating functions, the Binet formulas and some significant identities for these quaternions. Also, we give the matrix representations of the modified generalized Fibonacci and modified generalized Lucas quaternions.*

***Keywords:*** *Modified generalized Fibonacci sequence, modified generalized Lucas sequence, Recurrence Relations, Quaternions.*

## 1. INTRODUCTION

The Fibonacci numbers play an important role in various areas such as mathematics, physics, computer science and related fields. The Fibonacci numbers are defined by the recurrence relation

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n, \quad n \geq 0. \tag{1.1}$$

Another significant number sequence is the Lucas numbers which is defined by the recurrence relation

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n, \quad n \geq 0. \tag{1.2}$$

In recent years, there have been a lot of applications and generalizations of the Fibonacci and Lucas numbers in the literature [1-8]. For example, Falcon and Plaza[2] introduced the $k$ −Fibonacci sequence, $\{F_{k,n}\}_{n=0}^{\infty}$, by examining the recursive application of two geometrical transformations used in the (4TLE) partition. Yazlik and Taskara[4] defined the generalized $k$ −Horadam sequence and proved some properties of this sequence by means of determinant. In particular, Edson and Yayenie[5] presented an important generalization of the Fibonacci numbers, which is called as bi-periodic Fibonacci sequence, and then they derived the extended Binet formula, generating function and lots of identities of this sequence. Furthermore, Yayenie[6] defined the modified generalized Fibonacci sequence as

$$Q_0 = 0, \quad Q_1 = 1, \quad Q_n = \begin{cases} aQ_{n-1} + cQ_{n-2} & \text{if } n \text{ is even} \\ bQ_{n-1} + dQ_{n-2} & \text{if } n \text{ is odd} \end{cases}, \quad n \geq 2 \tag{1.3}$$

[1] Nevşehir Hacı Bektaş Veli University, Department of Mathematics, 50300 Nevşehir, Turkey.
E-mails: surekome@gmail.com; cahitkome@gmail.com; yyazlik@nevsehir.edu.tr.

where $a, b, c$ and $d$ are real numbers. Also he gave generating function, the generalized Binet formula and some basic identities for $Q_n$. Motivating by the studies [5] and [6], Bilgici [7] defined the bi-periodic Lucas numbers and modified generalized Lucas numbers and gave generating functions, the Binet formulas and some special identities for these sequences. He defined the modified generalized Lucas sequence as

$$U_0 = \frac{d+1}{d}, \quad U_1 = a, \quad U_n = \begin{cases} bU_{n-1} + dU_{n-2} & \text{if } n \text{ is even} \\ aU_{n-1} + cU_{n-2} & \text{if } n \text{ is odd} \end{cases}, \quad n \geq 2, \tag{1.4}$$

where $a, b, c$ and $d$ are real numbers. The generating functions of $Q_n$ and $U_n$ are given by

$$H(x) = \sum_{n=0}^{\infty} Q_n x^n = \frac{x(1+ax-cx^2)}{1-(ab+c+d)x^2+cdx^4} \tag{1.5}$$

and

$$U(x) = \sum_{n=0}^{\infty} U_n x^n = \frac{1}{d}\left(\frac{d+1+adx-(ab+cd+c)x^2+adx^3}{1-(ab+c+d)x^2+cdx^4}\right), \tag{1.6}$$

respectively. In addition, the Binet formulas of the sequences $Q_n$ and $U_n$ are also given by the following formulas:

$$Q_n = \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\left(\frac{\alpha^{\lfloor \frac{n}{2} \rfloor}(\alpha+d-c)^{n-\lfloor \frac{n}{2} \rfloor} - \beta^{\lfloor \frac{n}{2} \rfloor}(\beta+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta}\right) \tag{1.7}$$

and

$$U_n = \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}}\left(\frac{(\alpha+d+1)\alpha^{\lfloor \frac{n-1}{2} \rfloor}(\alpha+d-c)^{\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} - \frac{(\beta+d+1)\beta^{\lfloor \frac{n-1}{2} \rfloor}(\beta+d-c)^{\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta}\right) \tag{1.8}$$

where $\alpha = \frac{ab+c-d+\sqrt{(ab+c-d)^2+4abd}}{2}$ and $\beta = \frac{ab+c-d-\sqrt{(ab+c-d)^2+4abd}}{2}$ are the roots of the polynomial $x^2 - (ab+c-d)x - abd = 0$ and $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function that we use throughout the paper. Note that, we also assume that $\Delta = (ab+c-d)^2 + 4abd > 0$.

The quaternions, which arise in quantum mechanics, physics, mathematics, computer science and related areas, are a number system that extends the complex numbers. The quaternions were first introduced by William Rowan Hamilton in 1843. In general, a quaternion $q$ is defined by the following formula

$$q = q_0 + iq_1 + jq_2 + kq_3, \tag{1.9}$$

where $i, j, k$ are standart orthonormal basis in $\mathbb{R}^3$ and $q_0, q_1, q_2$ and $q_3$ are real numbers. In addition, the basis $i, j, k$ satisfy the following multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1.10}$$

Note that, the conjugate of the quaternion $\bar{q}$ is defined by

$$\overline{q} = q_0 - iq_1 - jq_2 - kq_3, \tag{1.11}$$

where $i, j, k$ satisfy the rules (1.10). Recently, the quaternions have attracted much attention of researchers [9-18]. For example, Horadam [9] defined the $nth$ Fibonacci and Lucas quaternions as

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3} \tag{1.12}$$

and

$$P_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3} \tag{1.13}$$

respectively. Ramirez [11] studied the combinatorial properties of the $k-$Fibonacci and $k-$Lucas quaternions by using the properties of the $k-$Fibonacci and $k-$Lucas numbers. Tan [12] proposed the bi-periodic Fibonacci quaternion $Q_n$ as

$$Q_n = \sum_{l=0}^{3} q_{n+l} e_l, \quad n \geq 0, \tag{1.14}$$

where $q_n$ is the bi-periodic Fibonacci sequence(see [5]). Tan et al. [13] also presented the bi-periodic Lucas quaternion $P_n$ as

$$P_n = \sum_{l=0}^{3} l_{n+l} e_l, \quad n \geq 0, \tag{1.15}$$

where $l_n$ is the bi-periodic Lucas numbers(see [7]), and then they obtained generating functions, the Binet formulas and some identities between bi-periodic Fibonacci and the bi-periodic Lucas quaternions.

In this paper, motivated by the studies [9-18], we present the generating functions, the Binet formulas, Catalan's identity, Cassini's identity and some new results for both the modified generalized Fibonacci and modified generalized Lucas quaternions, which are generalization of several studies in the literature such as [10-15], in Section 2 and 3. In the final section, we give a concise conclusion.

## 2. MODIFIED GENERALIZED FIBONACCI QUATERNIONS

In this section, we define the modified generalized Fibonacci quaternion $\Theta_n$. We give the generating function, the Binet formula and some significant identities for this quaternion.

**Definition 2.1.** For $\mathrm{n} \in \mathbb{N}_0$, the modified generalized Fibonacci quaternion $\Theta_\mathrm{n}$ is defined by

$$\Theta_n = \sum_{l=0}^{3} Q_{n+l} e_l, \tag{2.1}$$

where $Q_n$ is the $nth$ modified generalized Fibonacci numbers that is defined in (1.3). Moreover, Eq. (2.1) can be reduced to several quaternions for the special cases of $a, b, c$ and $d$.

For example,

- if we take $c = d = 1$, we obtain the bi-periodic Fibonacci quaternions in [12],
- if we take $c = d = 1$ and $a = b = k$, we obtain the $k$ −Fibonacci quaternions in [11],
- if we take $c = d = 1$ and $a = b = 2$, we obtain the Pell Fibonacci quaternions in [14],
- if we take $c = d = 2$ and $a = b = 1$, we obtain the Jacobsthal quaternions in [15] and
- if we take $c = d = 1$ and $a = b = 1$, we obtain the Fibonacci quaternions in [10].

**Theorem 2.2** The generating function for the modified generalized Fibonacci quaternion $\Theta_n$ is

$$G(t) = \frac{\Theta_0 + (\Theta_1 - b\Theta_0)t + (a-b)R_1(t) + (c-d)R_2(t)}{1 - bt - dt^2}, \tag{2.2}$$

where

$$R_1(t) = e_0 t f(t) + e_1(f(t) - t) + e_2\left(\frac{f(t)}{t} - 1\right) + e_3\left(\frac{f(t) - \left(t + (ab+d)t^3\right)}{t^2}\right),$$

$$R_2(t) = e_0 t^2 h(t) + e_1 t h(t) + e_2 h(t) + e_3\left(\frac{h(t) - at^2}{t}\right),$$

$$f(t) = \frac{t - ct^3}{1 - (ab + d + c)t^2 + cdt^4}$$

and

$$h(t) = \frac{at^2}{1 - (ab + d + c)t^2 + cdt^4}.$$

*Proof.* We use formal power series representation in order to find the generating function of $\Theta_m$. Now we define

$$G(t) = \sum_{m=0}^{\infty} \Theta_m t^m = \Theta_0 + \Theta_1 t + \sum_{m=2}^{\infty} \Theta_m t^m. \tag{2.3}$$

Note that,

$$btG(t) = \sum_{m=0}^{\infty} b\Theta_m t^{m+1} = \sum_{m=1}^{\infty} b\Theta_{m-1} t^m = bt\Theta_0 + \sum_{m=2}^{\infty} b\Theta_{m-1} t^m \tag{2.4}$$

and

$$dt^2 G(t) = \sum_{m=0}^{\infty} d\Theta_m t^{m+2} = \sum_{m=2}^{\infty} d\Theta_{m-2} t^m. \tag{2.5}$$

Since $Q_n$ satisfies the recurrence relations $Q_{2m} = aQ_{2m-1} + cQ_{2m-2}$ and $Q_{2m+1} = bQ_{2m} + dQ_{2m-1}$, we obtain

$$\begin{aligned}(1-bt-dt^2)G(t) &= \Theta_0+(\Theta_1-b\Theta_0)t+\sum_{m=2}^{\infty}(\Theta_m-b\Theta_{m-1}-d\Theta_{m-2})t^m\\ &= \Theta_0+(\Theta_1-b\Theta_0)t\\ &\quad +e_0\left((a-b)t\sum_{m=1}^{\infty}Q_{2m-1}t^{2m-1}+(c-d)t^2\sum_{m=1}^{\infty}Q_{2m-2}t^{2m-2}\right)\\ &\quad +e_1\left((a-b)\sum_{m=2}^{\infty}Q_{2m-1}t^{2m-1}+(c-d)t\sum_{m=2}^{\infty}Q_{2m-2}t^{2m-2}\right)\\ &\quad +e_2\left(\left(\frac{a-b}{t}\right)\sum_{m=2}^{\infty}Q_{2m-1}t^{2m-1}+(c-d)\sum_{m=2}^{\infty}Q_{2m-2}t^{2m-2}\right)\\ &\quad +e_3\left(\left(\frac{a-b}{t^2}\right)\sum_{m=3}^{\infty}Q_{2m-1}t^{2m-1}+\left(\frac{c-d}{t}\right)\sum_{m=3}^{\infty}Q_{2m-2}t^{2m-2}\right)\\ &= \Theta_0+(\Theta_1-b\Theta_0)t+e_0\big((a-b)tf(t)+(c-d)t^2h(t)\big)\\ &\quad +e_1\big((a-b)(f(t)-t)+(c-d)th(t)\big)\\ &\quad +e_2\left(\left(\frac{a-b}{t}\right)(f(t)-t)+(c-d)h(t)\right)\\ &\quad +e_3\left(\left(\frac{a-b}{t^2}\right)(f(t)-t-(ab+d)t^3)+\left(\frac{c-d}{t}\right)(h(t)-at^2)\right)\\ &= \Theta_0+(\Theta_1-b\Theta_0)t+(a-b)R_1(t)+(c-d)R_2(t),\end{aligned}$$

where

$$R_1(t)=e_0tf(t)+e_1(f(t)-t)+e_2\left(\frac{f(t)}{t}-1\right)+e_3\left(\frac{f(t)-\big(t+(ab+d)t^3\big)}{t^2}\right),$$

$$R_2(t)=e_0t^2h(t)+e_1th(t)+e_2h(t)+e_3\left(\frac{h(t)-at^2}{t}\right),$$

$$f(t)=\sum_{m=1}^{\infty}Q_{2m-1}t^{2m-1}\quad\text{and}\quad h(t)=\sum_{m=1}^{\infty}Q_{2m-2}t^{2m-2}.$$

On the other hand, the modified generalized Fibonacci numbers satisfy

$$\begin{aligned}Q_{2m-1} &= bQ_{2m-2}+dQ_{2m-3}\\ &= b(aQ_{2m-3}+cQ_{2m-4})+dQ_{2m-3}\\ &= (ab+d)Q_{2m-3}+bcQ_{2m-4}\\ &= (ab+d)Q_{2m-3}+cQ_{2m-3}-cdQ_{2m-5}\\ &= (ab+d+c)Q_{2m-3}-cdQ_{2m-5},\end{aligned}\tag{2.6}$$

and

$$\begin{aligned}Q_{2m-2} &= aQ_{2m-3}+cQ_{2m-4}\\ &= a(bQ_{2m-4}+dQ_{2m-5})+cQ_{2m-4}\\ &= (ab+c)Q_{2m-4}+adQ_{2m-5}\\ &= (ab+c)Q_{2m-4}+dQ_{2m-4}-cdQ_{2m-6}\\ &= (ab+d+c)Q_{2m-4}-cdQ_{2m-6}.\end{aligned}\tag{2.7}$$

Using (2.6) and (2.7), we obtain

$$\begin{aligned}&(1-(ab+d+c)t^2+cdt^4)f(t)\\ &\quad = t+(ab+d)t^3-(ab+d+c)t^3\\ &\qquad +\sum_{m=3}^{\infty}(Q_{2m-1}-(ab+d+c)Q_{2m-3}+cdQ_{2m-5})t^{2m-1},\end{aligned}$$

and

$$\begin{aligned}&(1-(ab+d+c)t^2+cdt^4)h(t)\\&\quad = at^2+\sum_{m=3}^{\infty}\left(Q_{2m-2}-(ab+d+c)Q_{2m-4}+cdQ_{2m-6}\right)t^{2m-2}.\end{aligned}$$

Rearranging the above expressions, we get

$$f(t)=\frac{t-ct^3}{1-(ab+d+c)t^2+cdt^4} \text{ and } h(t)=\frac{at^2}{1-(ab+d+c)t^2+cdt^4}.$$

Therefore, by using $f(t)$, $h(t)$, $R_1(t)$ and $R_2(t)$, we obtain the generating function of $\Theta_n$ as:

$$G(t)=\frac{\Theta_0+(\Theta_1-b\Theta_0)t+(a-b)R_1(t)+(c-d)R_2(t)}{1-bt-dt^2}. \tag{2.8}$$

Now, we derive the Binet formula of the modified generalized Fibonacci quaternion by the help of the Binet formula of $Q_n$.

**Theorem 2.3** For n $\in \mathbb{N}_0$, the Binet formula for the modified generalized Fibonacci quaternion is

$$\Theta_n=\frac{1}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\frac{\alpha_{\xi(n)}\alpha^{\lfloor\frac{n}{2}\rfloor}(\alpha+d-c)^{n-\lfloor\frac{n}{2}\rfloor}-\beta_{\xi(n)}\beta^{\lfloor\frac{n}{2}\rfloor}(\beta+d-c)^{n-\lfloor\frac{n}{2}\rfloor}}{\alpha-\beta}, \tag{2.9}$$

where

$$\alpha_{\xi(n)}=\sum_{l=0}^{3}\frac{a^{\xi(l+1-\xi(n))}}{(ab)^{\lfloor\frac{l+\xi(n)}{2}\rfloor}}(\alpha+d-c)^{\lfloor\frac{l+1-\xi(n)}{2}\rfloor}\alpha^{\lfloor\frac{l+\xi(n)}{2}\rfloor}e_l$$

and

$$\beta_{\xi(n)}=\sum_{l=0}^{3}\frac{a^{\xi(l+1-\xi(n))}}{(ab^{\lfloor\frac{l+\xi(n)}{2}\rfloor}}(\beta+d-c)^{\lfloor\frac{l+1-\xi(n)}{2}\rfloor}\beta^{\lfloor\frac{l+\xi(n)}{2}\rfloor}e_l.$$

*Proof.* By using the Binet formula of the modified generalized Fibonacci sequence, we can write

$$\begin{aligned}\Theta_{2n}&=\sum_{l=0}^{3}Q_{2n+l}e_l\\&=e_0\frac{a}{(ab)^n}\left(\frac{\alpha^n(\alpha+d-c)^n-\beta^n(\beta+d-c)^n}{\alpha-\beta}\right)\\&\quad+e_1\frac{1}{(ab)^n}\left(\frac{\alpha^n(\alpha+d-c)^{n+1}-\beta^n(\beta+d-c)^{n+1}}{\alpha-\beta}\right)\\&\quad+e_2\frac{a}{(ab)^{n+1}}\left(\frac{\alpha^{n+1}(\alpha+d-c)^{n+1}-\beta^{n+1}(\beta+d-c)^{n+1}}{\alpha-\beta}\right)\\&\quad+e_3\frac{1}{(ab)^{n+1}}\left(\frac{\alpha^{n+1}(\alpha+d-c)^{n+2}-\beta^{n+1}(\beta+d-c)^{n+2}}{\alpha-\beta}\right)\\&=\frac{1}{(ab)^n}\frac{\alpha^n(\alpha+d-c)^n}{\alpha-\beta}\Big(e_0a+e_1(\alpha+d-c)+e_2\left(\frac{a\alpha(\alpha+d-c)}{ab}\right)\\&\quad+e_3\left(\frac{\alpha(\alpha+d-c)^2}{ab}\right)\Big)-\frac{1}{(ab)^n}\frac{\beta^n(\beta+d-c)^n}{\alpha-\beta}\times(e_0a+\end{aligned}$$

$$+e_1(\beta+d-c)+e_2\left(\frac{\alpha\beta(\beta+d-c)}{ab}\right)+e_3\left(\frac{\beta(\beta+d-c)^2}{ab}\right))$$

$$=\frac{1}{(ab)^n}\frac{\alpha_0\alpha^n(\alpha+d-c)^n-\beta_0\beta^n(\beta+d-c)^n}{\alpha-\beta}, \tag{2.10}$$

where

$$\alpha_0=\sum_{l=0}^{3}\frac{a^{\xi(l+1)}}{(ab)^{\left\lfloor\frac{l}{2}\right\rfloor}}(\alpha+d-c)^{\left\lfloor\frac{l+1}{2}\right\rfloor}\alpha^{\left\lfloor\frac{l}{2}\right\rfloor}e_l$$

and

$$\beta_0=\sum_{l=0}^{3}\frac{a^{\xi(l+1)}}{(ab)^{\left\lfloor\frac{l}{2}\right\rfloor}}(\beta+d-c)^{\left\lfloor\frac{l+1}{2}\right\rfloor}\beta^{\left\lfloor\frac{l}{2}\right\rfloor}e_l.$$

Similarly, we can obtain

$$\Theta_{2n+1}=\frac{1}{(ab)^n}\frac{\alpha_1\alpha^n(\alpha+d-c)^{n+1}-\beta_1\beta^n(\beta+d-c)^{n+1}}{\alpha-\beta}, \tag{2.11}$$

where

$$\alpha_1=\sum_{l=0}^{3}\frac{a^{\xi(l)}}{(ab)^{\left\lfloor\frac{l+1}{2}\right\rfloor}}(\alpha+d-c)^{\left\lfloor\frac{l}{2}\right\rfloor}\alpha^{\left\lfloor\frac{l+1}{2}\right\rfloor}e_l$$

and

$$\beta_1=\sum_{l=0}^{3}\frac{a^{\xi(l)}}{(ab)^{\left\lfloor\frac{l+1}{2}\right\rfloor}}(\beta+d-c)^{\left\lfloor\frac{l}{2}\right\rfloor}\beta^{\left\lfloor\frac{l+1}{2}\right\rfloor}e_l.$$

Combining the equations (2.10) and (2.11), we get

$$\Theta_n=\frac{1}{(ab)^{\left\lfloor\frac{n}{2}\right\rfloor}}\frac{\alpha_{\xi(n)}\alpha^{\left\lfloor\frac{n}{2}\right\rfloor}(\alpha+d-c)^{n-\left\lfloor\frac{n}{2}\right\rfloor}-\beta_{\xi(n)}\beta^{\left\lfloor\frac{n}{2}\right\rfloor}(\beta+d-c)^{n-\left\lfloor\frac{n}{2}\right\rfloor}}{\alpha-\beta},$$

where

$$\alpha_{\xi(n)}=\sum_{l=0}^{3}\frac{a^{\xi(l+1-\xi(n))}}{(ab)^{\left\lfloor\frac{l+\xi(n)}{2}\right\rfloor}}(\alpha+d-c)^{\left\lfloor\frac{l+1-\xi(n)}{2}\right\rfloor}\alpha^{\left\lfloor\frac{l+\xi(n)}{2}\right\rfloor}e_l$$

and

$$\beta_{\xi(n)}=\sum_{l=0}^{3}\frac{a^{\xi(l+1-\xi(n))}}{(ab)^{\left\lfloor\frac{l+\xi(n)}{2}\right\rfloor}}(\beta+d-c)^{\left\lfloor\frac{l+1-\xi(n)}{2}\right\rfloor}\beta^{\left\lfloor\frac{l+\xi(n)}{2}\right\rfloor}e_l.$$

In the following theorem we derive the Catalan's identity with the help of the Binet formula of $Q_n$. Furthermore, we give the Cassini's identity which is the special case of the Catalan's identity for $r = 1$.

**Theorem 2.4** *(Catalan's identity) For $n, r \in \mathbb{N}_0$ and $r \leq n$, we have the identity*

$$\begin{aligned}\Theta_{2(n+r)+\xi(i)}\Theta_{2(n-r)+\xi(i)} - \Theta_{2n+\xi(i)}^2 &= \frac{(-c)^{\xi(i)}}{(ab)^{2r}(\alpha-\beta)^2}\Big[\alpha_{\xi(i)}\beta_{\xi(i)}\Big((ab)^{2r+\xi(i)}(cd)^n - (ab)^{2r+\xi(i)}(cd)^n\left(\frac{\alpha+d}{\beta+d}\right)^r\Big)\\ &+\beta_{\xi(i)}\alpha_{\xi(i)}\Big((ab)^{2r+\xi(i)}(cd)^n - (ab)^{2r+\xi(i)}(cd)^n\left(\frac{\beta+d}{\alpha+d}\right)^r\Big)\Big],\end{aligned}$$

where $\alpha_{\xi(i)}$ and $\beta_{\xi(i)}$ are defined in Theorem 2.3 and $i \in \{0,1\}$.

*Proof.* By using the Binet formula of the modified generalized Fibonacci quaternion, for $i = 0$, we get

$$\begin{aligned}\Theta_{2(n+r)}\Theta_{2(n-r)} - \Theta_{2n}^2 &= \left(\frac{1}{(ab)^{n+r}}\frac{\alpha_0\alpha^{n+r}(\alpha+d-c)^{n+r} - \beta_0\beta^{n+r}(\beta+d-c)^{n+r}}{\alpha-\beta}\right)\\ &\times\left(\frac{1}{(ab)^{n-r}}\frac{\alpha_0\alpha^{n-r}(\alpha+d-c)^{n-r} - \beta_0\beta^{n-r}(\beta+d-c)^{n-r}}{\alpha-\beta}\right)\\ &-\left(\frac{1}{(ab)^{n}}\frac{\alpha_0\alpha^{n}(\alpha+d-c)^{n} - \beta_0\beta^{n}(\beta+d-c)^{n}}{\alpha-\beta}\right)^2\\ &= \frac{1}{(ab)^{2n}(\alpha-\beta)^2}\big(\alpha_0\beta_0(\alpha^n\beta^n(\alpha+d-c)^n(\beta+d-c)^n\\ &-\alpha^{n+r}\beta^{n-r}(\alpha+d-c)^{n+r}(\beta+d-c)^{n-r})\\ &+\beta_0\alpha_0(\alpha^n\beta^n(\alpha+d-c)^n(\beta+d-c)^n\\ &-\alpha^{n-r}\beta^{n+r}(\alpha+d-c)^{n-r}(\beta+d-c)^{n+r})\big)\\ &= \frac{1}{(ab)^{2r}(\alpha-\beta)^2}\Big(\alpha_0\beta_0\Big((ab)^{2r}(cd)^n - (ab)^{2r}(cd)^n\left(\frac{\alpha+d}{\beta+d}\right)^r\Big)\\ &+\beta_0\alpha_0\Big((ab)^{2r}(cd)^n - (ab)^{2r}(cd)^n\left(\frac{\beta+d}{\alpha+d}\right)^r\Big)\Big).\end{aligned} \tag{2.12}$$

Similarly, for $i = 1$, we get

$$\begin{aligned}\Theta_{2(n+r)+1}\Theta_{2(n-r)+1} - \Theta_{2n+1}^2 &= -\frac{c}{(ab)^{2r}(\alpha-\beta)^2}\Big(\alpha_1\beta_1\Big((ab)^{2r+1}(cd)^n - (ab)^{2r+1}(cd)^n\left(\frac{\alpha+d}{\beta+d}\right)^r\Big)\\ &+\beta_1\alpha_1\Big((ab)^{2r+1}(cd)^n - (ab)^{2r+1}(cd)^n\left(\frac{\beta+d}{\alpha+d}\right)^r\Big)\Big).\end{aligned} \tag{2.13}$$

By combining the equations (2.12) and (2.13), we obtain

$$\begin{aligned}\Theta_{2(n+r)+\xi(i)}\Theta_{2(n-r)+\xi(i)} - \Theta_{2n+\xi(i)}^2 &= \frac{(-c)^{\xi(i)}}{(ab)^{2r}(\alpha-\beta)^2}\Big[\alpha_{\xi(i)}\beta_{\xi(i)}\Big((ab)^{2r+\xi(i)}(cd)^n - (ab)^{2r+\xi(i)}(cd)^n\left(\frac{\alpha+d}{\beta+d}\right)^r\Big)\\ &+\beta_{\xi(i)}\alpha_{\xi(i)}\Big((ab)^{2r+\xi(i)}(cd)^n - (ab)^{2r+\xi(i)}(cd)^n\left(\frac{\beta+d}{\alpha+d}\right)^r\Big)\Big],\end{aligned}$$

where $\alpha_{\xi(i)}$ and $\beta_{\xi(i)}$ are defined in Theorem 2.3 and $i \in \{0,1\}$.

**Corollary 2.5** *(Cassini's identity) For $n \in \mathbb{N}_0$, we have the identity*

$$\Theta_{2(n+1)+\xi(i)}\Theta_{2(n-1)+\xi(i)} - \Theta_{2n+\xi(i)}^2$$

$$= \frac{(-c)^{\xi(i)}}{(ab)^2(\alpha-\beta)^2}[\alpha_{\xi(i)}\beta_{\xi(i)}((ab)^{2+\xi(i)}(cd)^n - (ab)^{2+\xi(i)}(cd)^n\left(\frac{\alpha+d}{\beta+d}\right)) + \beta_{\xi(i)}\alpha_{\xi(i)}((ab)^{2+\xi(i)}(cd)^n - (ab)^{2+\xi(i)}(cd)^n\left(\frac{\beta+d}{\alpha+d}\right))],$$

where $\alpha_{\xi(i)}$ and $\beta_{\xi(i)}$ are defined in Theorem 2.3 and $i \in \{0,1\}$.

The matrix representation of the modified generalized Fibonacci quaternion can be given in the following theorem.

**Theorem 2.6** Let $n \geq 1$ be integer. Then

$$\begin{pmatrix} \Theta_{2n+i} & \Theta_{2(n-1)+i} \\ \Theta_{2(n+1)+i} & \Theta_{2n+i} \end{pmatrix} = \begin{pmatrix} \Theta_{2+i} & \Theta_i \\ \Theta_{4+i} & \Theta_{2+i} \end{pmatrix}\begin{pmatrix} ab+c+d & 1 \\ -cd & 0 \end{pmatrix}^{n-1}, \tag{2.14}$$

where $i \in \{0,1\}$.

*Proof.* We prove the theorem by induction on $n$. If $n = 1$ then the result is clear. Now we assume that, for any integer $k$ such as $1 \leq k \leq n$,

$$\begin{pmatrix} \Theta_{2k+i} & \Theta_{2(k-1)+i} \\ \Theta_{2(k+1)+i} & \Theta_{2k+i} \end{pmatrix} = \begin{pmatrix} \Theta_{2+i} & \Theta_i \\ \Theta_{4+i} & \Theta_{2+i} \end{pmatrix}\begin{pmatrix} ab+c+d & 1 \\ -cd & 0 \end{pmatrix}^{k-1}.$$

Then, for $n = k+1$, by using the identities (2.6) and (2.7) we get

$$\begin{aligned} &\begin{pmatrix} \Theta_{2+i} & \Theta_i \\ \Theta_{4+i} & \Theta_{2+i} \end{pmatrix}\begin{pmatrix} ab+c+d & 1 \\ -cd & 0 \end{pmatrix}^{k} \\ &= \begin{pmatrix} \Theta_{2+i} & \Theta_i \\ \Theta_{4+i} & \Theta_{2+i} \end{pmatrix}\begin{pmatrix} ab+c+d & 1 \\ -cd & 0 \end{pmatrix}^{k-1}\begin{pmatrix} ab+c+d & 1 \\ -cd & 0 \end{pmatrix} \\ &= \begin{pmatrix} \Theta_{2k+i} & \Theta_{2(k-1)+i} \\ \Theta_{2(k+1)+i} & \Theta_{2k+i} \end{pmatrix}\begin{pmatrix} ab+c+d & 1 \\ -cd & 0 \end{pmatrix} \\ &= \begin{pmatrix} \Theta_{2(k+1)+i} & \Theta_{2k+i} \\ \Theta_{2(k+2)+i} & \Theta_{2(k+1)+i} \end{pmatrix}, \end{aligned}$$

where $i \in \{0,1\}$. Therefore, the proof is completed.

## 3. MODIFIED GENERALIZED LUCAS QUATERNIONS

In this section, we define the modified generalized Lucas quaternion $\vartheta_n$. We give the generating function, the Binet formula and some important identities for this quaternion. The theorems and results in this section can be proven similar to the results in Section 2. Hence, we omit the proofs.

**Definition 3.1** For $n \in \mathbb{N}_0$, the modified generalized Lucas quaternion $\vartheta_n$ is defined by

$$\vartheta_n = \sum_{l=0}^{3} U_{n+l} e_l, \tag{3.1}$$

where $U_n$ is the modified generalized Lucas numbers that is defined in (1.4). Moreover, Eq. (3.1) can be reduced to several quaternions for the special cases of $a, b, c$ and $d$. For example,

- if we take $c = d = 1$, we obtain the bi-periodic Lucas quaternions in [13],
- if we take $c = d = 1$ and $a = b = k$, we obtain the $k$ −Lucas quaternions in [11],
- if we take $c = d = 1$ and $a = b = 1$, we obtain the Lucas quaternions in [10].

**Theorem 3.2** The generating function for the modified generalized Lucas quaternion $\vartheta_n$ is

$$L(t) = \frac{\vartheta_0 + (\vartheta_1 - a\vartheta_0)t + (b-a)R_1(t) + (d-c)R_2(t)}{1 - at - ct^2}, \tag{3.2}$$

where

$$R_1(t) = e_0 t f(t) + e_1(f(t) - at) + e_2\left(\frac{f(t)}{t} - a\right)$$
$$+ e_3\left(\frac{f(t) - at - (a^2b + ad + ac + a)t^3}{t^2}\right),$$

$$R_2(t) = e_0 t^2 h(t) + e_1 t\left(h(t) - \left(\frac{d+1}{d}\right)\right) + e_2\left(h(t) - \left(\frac{d+1}{d}\right)\right)$$
$$+ e_3\left(\frac{h(t) - \left(\frac{d+1}{d}\right) - (ab+d+1)t^2}{t}\right),$$

$$f(t) = \frac{at + at^3}{1 - (ab+d+c)t^2 + cdt^4}$$

and

$$h(t) = \frac{\left(\frac{d+1}{d}\right) + (ab+d+1)t^2 - (ab+d+c)\left(\frac{d+1}{d}\right)t^2}{1 - (ab+d+c)t^2 + cdt^4}.$$

**Theorem 3.3** For $n \in \mathbb{N}_0$, the Binet formula for the modified generalized Lucas quaternion is

$$\vartheta_n = \frac{1}{(ab)^{\left\lfloor \frac{n-1}{2} \right\rfloor}}\left(\frac{\alpha^*_{\xi(n)}(\alpha+d+1)\alpha^{\left\lfloor \frac{n-1}{2} \right\rfloor}(\alpha+d-c)^{\left\lfloor \frac{n}{2} \right\rfloor}}{\alpha - \beta} - \frac{\beta^*_{\xi(n)}(\beta+d+1)\beta^{\left\lfloor \frac{n-1}{2} \right\rfloor}(\beta+d-c)^{\left\lfloor \frac{n}{2} \right\rfloor}}{\alpha - \beta}\right), \tag{3.3}$$

where

$$\alpha^*_{\xi(n)} = \sum_{l=0}^{3} \frac{a^{\xi(l+\xi(n))}}{(ab)^{\left\lfloor \frac{l+1-\xi(n)}{2} \right\rfloor}} (\alpha + d - c)^{\left\lfloor \frac{l+\xi(n)}{2} \right\rfloor} \alpha^{\left\lfloor \frac{l+1-\xi(n)}{2} \right\rfloor} e_l$$

and

$$\beta^*_{\xi(n)} = \sum_{l=0}^{3} \frac{a^{\xi(l+\xi(n))}}{(ab)^{\left\lfloor \frac{l+1-\xi(n)}{2} \right\rfloor}} (\beta + d - c)^{\left\lfloor \frac{l+\xi(n)}{2} \right\rfloor} \beta^{\left\lfloor \frac{l+1-\xi(n)}{2} \right\rfloor} e_l.$$

In the following theorem we derive the Catalan's identity with the help of the Binet formula of $U_n$. Furthermore, we give the Cassini's identity which is the special case of the Catalan's identity for $r = 1$.

**Theorem 3.4** (Catalan's identity) For $n, r \in \mathbb{N}_0$ and $r \leq n$, we have the identity

$$\begin{aligned}\vartheta_{2(n+r)+\xi(i)}\vartheta_{2(n-r)+\xi(i)} - \vartheta_{2n+\xi(i)}^2 &= \frac{(-c)^{1-\xi(i)}(\alpha+d+1)(\beta+d+1)}{(ab)^{2r}(\alpha-\beta)^2} \\ &\times [\alpha_{\xi(i)}^*\beta_{\xi(i)}^*((ab)^{2r+1-\xi(i)}(cd)^{n-1+\xi(i)} \\ &-(ab)^{2r+1-\xi(i)}(cd)^{n-1+\xi(i)}\left(\frac{\alpha+d}{\beta+d}\right)^r) \\ &+\beta_{\xi(i)}^*\alpha_{\xi(i)}^*((ab)^{2r+1-\xi(i)}(cd)^{n-1+\xi(i)} \\ &-(ab)^{2r+1-\xi(i)}(cd)^{n-1+\xi(i)}\left(\frac{\beta+d}{\alpha+d}\right)^r)],\end{aligned}$$

where $\alpha_{\xi(i)}^*$ and $\beta_{\xi(i)}^*$ are defined in Theorem 3.3 and $i \in \{0,1\}$.

**Corollary 3.5** (Cassini's identity) For $n \in \mathbb{N}_0$, we have the identity

$$\begin{aligned}\vartheta_{2(n+1)+\xi(i)}\vartheta_{2(n-1)+\xi(i)} - \vartheta_{2n+\xi(i)}^2 &= \frac{(-c)^{1-\xi(i)}(\alpha+d+1)(\beta+d+1)}{(ab)^2(\alpha-\beta)^2} \times [\alpha_{\xi(i)}^*\beta_{\xi(i)}^*((ab)^{3-\xi(i)}(cd)^{n-1+\xi(i)} \\ &-(ab)^{3-\xi(i)}(cd)^{n-1+\xi(i)}\left(\frac{\alpha+d}{\beta+d}\right)) + \beta_{\xi(i)}^*\alpha_{\xi(i)}^*((ab)^{3-\xi(i)}(cd)^{n-1+\xi(i)} \\ &-(ab)^{3-\xi(i)}(cd)^{n-1+\xi(i)}\left(\frac{\beta+d}{\alpha+d}\right))],\end{aligned}$$

where $\alpha_{\xi(i)}^*$ and $\beta_{\xi(i)}^*$ are defined in Theorem 3.3 and $i \in \{0,1\}$.

**Theorem 3.6** Let $n \geq 1$ be integer. Then the modified generalized Lucas quaternion satisfies the relation

$$\vartheta_n = \Theta_{n-1} + \Theta_{n+1}. \tag{3.4}$$

*Proof.* By considering the identity $U_n = Q_{n-1} + Q_{n+1}$, which is given by the Theorem 20 in [7], we can easily obtain the desired result.

**Theorem 3.7** Let $n \geq 1$ be integer. Then

$$\begin{pmatrix}\vartheta_{2n+i} & \vartheta_{2(n-1)+i} \\ \vartheta_{2(n+1)+i} & \vartheta_{2n+i}\end{pmatrix} = \begin{pmatrix}\vartheta_{2+i} & \vartheta_i \\ \vartheta_{4+i} & \vartheta_{2+i}\end{pmatrix}\begin{pmatrix}ab+c+d & 1 \\ -cd & 0\end{pmatrix}^{n-1}, \tag{3.5}$$

where $i \in \{0,1\}$.

*Proof.* By considering the identities in Lemma 2 in [7], the proof can be made similar to the Theorem 2.6.

## 4. CONCLUSION

In this paper, we define the modified generalized Fibonacci and modified generalized Lucas quaternions. These quaternions are the generalization of several studies in the literature such as [10-15]. Moreover, we give the Catalan's identity, Cassini's identity and matrix representations in more general form. Since our study both generalization of several studies in the literature and includes some new results, it contributes to the literature by providing essential information on the generalization of the quaternions.

## REFERENCES


[1] Koshy, T., *Fibonacci and Lucas numbers with applications*, A Wiley-Interscience Publication, 2001.
[2] Falcón, S., Plaza A., *Chaos, Solitons & Fractals*, **33**(1), 38, 2007.
[3] Falcón, S., *International Journal of Contemporary Mathematical Sciences*, **6**(21), 1039, 2011.
[4] Yazlik, Y., Taskara, N., *Computers & Mathematics with Applications*, **63**(1), 36, 2012.
[5] Edson, M., Yayenie, O., *Integers*, **9**(6), 639, 2009.
[6] Yayenie, O., *Applied Mathematics and Computation*, **217**(12), 5603, 2011.
[7] Bilgici, G., *Applied Mathematics and Computation*, **245**, 526, 2014.
[8] Yazlik, Y., Köme, C., Madhusudanan V., *Journal of Computational Analysis and Applications*, **25**(4), 657, 2018.
[9] Horadam, A.F., *The American Mathematical Monthly*, **70**(3), 289, 1963.
[10] Halici, S., *Advances in applied Clifford Algebras*, **22**(2), 321, 2012.
[11] Ramirez, J.L., *Analele stiintifice ale Universitatii Ovidius Constanta*, **23**(2), 201, 2015.
[12] Tan, E., *Chaos, Solitons & Fractals*, **82**, 1, 2016.
[13] Tan, E., Yilmaz, S., Sahin, M., *Chaos, Solitons & Fractals*, **85**, 138, 2016.
[14] Çimen, C.B., İpek, A., *Advances in Applied Clifford Algebras*, **26**(1), 39, 2016.
[15] Szynal-Liana, A., Wloch, I., *Advances in Applied Clifford Algebras*, **26**(1), 441, 2016.
[16] Halici, S., *Advances in applied Clifford Algebras*, **23**(1), 105, 2013.
[17] Catarino, P., *Chaos, Solitons & Fractals*, **77**, 1, 2015.
[18] Halici, S., Karatas, A., *Chaos, Solitons & Fractals*, **98**, 178, 2017.