Global behavior of two-dimensional difference equations system with two period coefficients

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Abstract

In this paper, we investigate the following system of difference equations

$$x_{n+1}=\frac{\alpha_n}{1+y_nx_{n-1}},\ y_{n+1}=\frac{\beta_n}{1+x_ny_{n-1}},\ n\in\mathbb{N}_0,$$

where the sequences $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$ are positive, real and periodic with period two and the initial values x_{-1}, x_0, y_{-1}, y_0 are non-negative real numbers. We show that every positive solution of the system is bounded and examine their global behaviors. In addition, we give closed forms of the general solutions of the system by using the change of variables. Finally, we present a numerical example to support our results.

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1 Introduction

As a prototype, in [1], Drymonis investigated the global stability, periodic character and boundedness of solution of the following difference equation by distinguishing several special cases

$$x_{n+1}=\frac{\alpha_n+\beta_nx_nx_{n-1}+\gamma_nx_{n-1}}{A_n+B_nx_nx_{n-1}+C_nx_{n-1}},\quad n\in\mathbb{N}_0,\tag{1}$$

where the parameters α_n, β_n, γ_n, A_n, B_n, C_n are non-negative periodic sequences and the initial values x_{-1}, x_0 are non-negative real numbers. In [2–5], equation (1) with constant coefficients is studied in the global stability, periodic and boundedness of solutions of some particular cases. Kulenovic et al, obtained five equations for the related of equation (1) with constant coefficients in [5]. Moreover, Amleh et al. studied thirty equations which are special case of equation (1) and constant coefficients in [2, 3]. One of thirty equations considered in [2] is the rational difference equation given as follows:

$$x_{n+1}=\frac{\alpha}{1+x_nx_{n-1}},\ n\in\mathbb{N}_0.\tag{2}$$

Further, some featured studies on the stability of the particular cases with constant coefficients of equation (1) can be found in the literature, (see, [6–11]). On the other hand, many authors obtain some closed-form formulas which are solutions special cases of the equation (1) in [4,12–19]. The interesting thing is that all of them have constant coefficients. Equation (1) is extended to the two-dimensional and the three-dimensional systems of difference equation with constant coefficients

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and obtained in the closed form the solutions in [20–34]. In addition, the global stability of the system of extending of equation (1) with constant coefficients is studied in [35–37].
According to the mentioned literature, there is no particular case using variable coefficients with the system of equation (1). Motivated by this, we extend equation (2) to the system of difference equations with periodic coefficients as follows:

$$x_{n+1} = \frac{\alpha_n}{1 + y_n x_{n-1}},\ y_{n+1} = \frac{\beta_n}{1 + x_n y_{n-1}},\ n \in \mathbb{N}_0, \tag{3}$$

where the sequences $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$ are positive, real and periodic with period two and the initial values x_{-1}, x_0, y_{-1}, y_0 are non-negative real numbers. Firstly, we show that every positive solution of the system (3) is bounded and then state the global behavior of positive solution of the system (3). We also give closed forms of the general solutions of the system (3) by using change of variables. Finally, we present a numerical example to support effective results.

Throughout this paper, we use the following sequences $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$,

$$\alpha_n = \begin{cases} a_1, & \text{if } n \text{ is even,} \\ b_1, & \text{if } n \text{ is odd} \end{cases} \text{ and } a_1 > 0,\ b_1 > 0,\ a_1 \neq b_1,$$

$$\beta_n = \begin{cases} a_2, & \text{if } n \text{ is even,} \\ b_2, & \text{if } n \text{ is odd} \end{cases} \text{ and } a_2 > 0,\ b_2 > 0,\ a_2 \neq b_2.$$

Then the system (3) can be written as follows:

$$x_{2n+1} = \frac{a_1}{1 + y_{2n} x_{2n-1}}, \quad x_{2n+2} = \frac{b_1}{1 + y_{2n+1} x_{2n}}, \tag{4}$$

$$y_{2n+1} = \frac{a_2}{1 + x_{2n} y_{2n-1}}, \quad y_{2n+2} = \frac{b_2}{1 + x_{2n+1} y_{2n}}. \tag{5}$$

To conduct the stability analysis, we assume that

$$x_{2n-1} = u_n,\ x_{2n} = v_n,\ y_{2n-1} = w_n,\ y_{2n} = t_n,\ n \in \mathbb{N}_0. \tag{6}$$

Thus (4) and (5) are obtained in the following form:

$$\begin{cases} u_{n+1} = \frac{a_1}{1+t_n u_n} \\ v_{n+1} = \frac{b_1(1+v_n w_n)}{1+v_n w_n + a_2 v_n} \\ w_{n+1} = \frac{a_2}{1+v_n w_n} \\ t_{n+1} = \frac{b_2(1+t_n u_n)}{1+t_n u_n + a_1 t_n} \end{cases}, \quad n \in \mathbb{N}_0. \tag{7}$$

We conclude that the system (7) is equivalent to the system (3). From now on we will use the system (7) instead of the system (3). Note that the following equations

$$\begin{cases} u_{n+1} = \frac{a_1}{1+t_n u_n} \\ t_{n+1} = \frac{b_2(1+t_n u_n)}{1+t_n u_n + a_1 t_n} \end{cases} \tag{8}$$

are independent from (v_n, w_n) and

$$\begin{cases} v_{n+1} = \frac{b_1(1+v_n w_n)}{1+v_n w_n + a_2 v_n} \\ w_{n+1} = \frac{a_2}{1+v_n w_n} \end{cases} \tag{9}$$

are also independent from (u_n, t_n). This means that the system (8) and the system (9) are the two-dimensional systems of difference equations. We see that if $\left(\overline{u}, \overline{v}, \overline{w}, \overline{t}\right)$ is an equilibrium point of the system (7), then the corresponding equilibrium points of (8) and (9) are $\left(\overline{u}, \overline{t}\right)$ and $(\overline{v}, \overline{w})$, respectively.

Now, we give some results concerning difference equations.

Lemma 1.1. [38] Consider the system $u_{n+1} = f(u_n, v_n)$, $v_{n+1} = g(u_n, v_n)$, $n \in \mathbb{N}_0$. Let $F = (f, g)$ be a continuously differentiable function defined on an open set $D \subset \mathbb{R} \times \mathbb{R}$.

(a) If the eigenvalues of the Jacobian matrix $J_F(\overline{u}, \overline{v})$, that is, both roots of its characteristic equation

$$\lambda^2 - T_r J_F(\overline{u}, \overline{v})\lambda + Det J_F(\overline{u}, \overline{v}) = 0, \tag{10}$$

lie inside the unit disk, then the equilibrium point $(\bar{u}, \bar{v})$ of the system $u_{n+1} = f(u_n, v_n)$, $v_{n+1} = g(u_n, v_n)$ is locally asymptotically stable.

(b) A necessary and sufficient condition for both roots of equation (10) to lie inside the unit disk is

$$|T_r J_F(\overline{u}, \overline{v})| < 1 + Det J_F(\overline{u}, \overline{v}) < 2.$$

Lemma 1.2. [39] Consider the cubic equation

$$P(z) = z^3 - \alpha z^2 - \beta z - \gamma = 0. \tag{11}$$

The equation (11) has the discriminant

$$\Delta = -\alpha^2\beta^2 - 4\beta^3 + 4\alpha^3\gamma + 27\gamma^2 + 18\alpha\beta\gamma. \tag{12}$$

Thus the following statements are true;

(i) If $\Delta < 0$ then the polynomial P has three distinct real zeros ρ_1, ρ_2, ρ_3.

(ii) If $\Delta = 0$ then there are two sub cases:

- **(a)** If $\beta = \frac{-\alpha^2}{3}$ and $\gamma = \frac{\alpha^3}{27}$, then the polynomial P has the triple root $\rho = \frac{\alpha}{3}$,
- **(b)** If $\beta \neq \frac{-\alpha^2}{3}$ or $\gamma \neq \frac{\alpha^3}{27}$, then the polynomial P has the double root r and the simple root ρ.

(iii) If $\Delta > 0$ then the polynomial P has one real root p and two complex roots $re^{\pm i\theta}$, $\theta \in (0, \pi)$.

2 Main results

In this section, we prove our main results.

Lemma 2.1. Assume that $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$ are positive periodic sequences of prime period 2. Then every positive solution of the system (3) is bounded.

Proof. From the system (3), we have

$$x_{n+1}=\frac{\alpha_n}{1+y_n x_{n-1}}\le \alpha_n,\quad y_{n+1}=\frac{\beta_n}{1+x_n y_{n-1}}\le \beta_n, \tag{13}$$

for $n\in\mathbb{N}_0$. Then we see that $x_{2n+1}\le a_1$, $x_{2n+2}\le b_1$, $y_{2n+1}\le a_2$ and $y_{2n+2}\le b_2$, for $n\in\mathbb{N}_0$. Combining (3) and (13), we have

$$x_{2n+1}=\frac{\alpha_{2n}}{1+y_{2n}x_{2n-1}}\ge\frac{\alpha_{2n}}{1+b_2a_1},\quad x_{2n+2}=\frac{\alpha_{2n+1}}{1+y_{2n+1}x_{2n}}\ge\frac{\alpha_{2n+1}}{1+a_2b_1},$$

$$y_{2n+1}=\frac{\beta_{2n}}{1+x_{2n}y_{2n-1}}\ge\frac{\beta_{2n}}{1+b_1a_2},\quad y_{2n+2}=\frac{\beta_{2n+1}}{1+x_{2n+1}y_{2n}}\ge\frac{\beta_{2n+1}}{1+a_1b_2},$$

for $n\in\mathbb{N}_0$. Consequently, we get

$$\frac{a_1}{1+b_2a_1}\le x_{2n+1}\le a_1,\quad \frac{b_1}{1+a_2b_1}\le x_{2n+2}\le b_1, \tag{14}$$

$$\frac{a_2}{1+a_2b_1}\le y_{2n+1}\le a_2,\quad \frac{b_2}{1+a_1b_2}\le y_{2n+2}\le b_2, \tag{15}$$

for $n\in\mathbb{N}_0$. ∎

2.1 Locally Asymptotically Stability

In this subsection, we study locally asymptotically stability of the unique positive equilibrium $\left(\overline{u},\bar{t},\overline{w},\overline{v}\right)=\left(\overline{u},\frac{b_2}{a_1}\overline{u},\overline{w},\frac{b_1}{a_2}\overline{w}\right)$ of the system (7).

Lemma 2.2. The system (7) has the unique positive equilibrium point on $\left(\frac{a_1}{1+a_1b_2},a_1\right)\times\left(\frac{b_2}{1+a_1b_2},b_2\right)\times\left(\frac{a_2}{1+a_2b_1},a_2\right)\times\left(\frac{b_1}{1+a_2b_1},b_1\right)$.

Proof. The equilibrium points of the system (7) are the solutions of the algebraic systems

$$\overline{u}=\frac{a_1}{1+\bar{t}\overline{u}},\quad \overline{v}=\frac{b_1\left(1+\overline{v}\overline{w}\right)}{1+\overline{v}\overline{w}+a_2\overline{v}},\quad \overline{w}=\frac{a_2}{1+\overline{v}\overline{w}},\quad \bar{t}=\frac{b_2\left(1+\bar{t}\overline{u}\right)}{1+\bar{t}\overline{u}+a_1\bar{t}}. \tag{16}$$

From (16), we obtain the following equalities:

$$\overline{v}=\frac{b_1}{a_2}\overline{w},\quad \bar{t}=\frac{b_2}{a_1}\overline{u}. \tag{17}$$

Substituting (17) into (16), we have the polynomial equations

$$P\left(\overline{u}\right)=\overline{u}^3+\frac{a_1}{b_2}\overline{u}-\frac{a_1^2}{b_2}=0,\quad R\left(\overline{w}\right)=\overline{w}^3+\frac{a_2}{b_1}\overline{w}-\frac{a_2^2}{b_1}=0. \tag{18}$$

From (6), (14), (15) and (18), we have

$$P(a_1) = a_1^3 > 0, \quad R(a_2) = a_2^3 > 0 \tag{19}$$

and

$$\begin{aligned} P\left(\frac{a_1}{1+a_1b_2}\right) &= -\frac{a_1^3\left((1+a_1b_2)^2-1\right)}{(1+a_1b_2)^3} < 0, \\ R\left(\frac{a_2}{1+a_2b_1}\right) &= -\frac{a_2^3\left((1+a_2b_1)^2-1\right)}{(1+a_2b_1)^3} < 0. \end{aligned} \tag{20}$$

Since

$$P'(\overline{u}) = 3\overline{u}^2 + \frac{a_1}{b_2} > 0, \quad R'(\overline{w}) = 3\overline{w}^2 + \frac{a_2}{b_1} > 0, \tag{21}$$

$P(\overline{u})$ has the unique zero on $\left(\frac{a_1}{1+a_1b_2}, a_1\right)$ and $R(\overline{w})$ has the unique zero on $\left(\frac{a_2}{1+a_2b_1}, a_2\right)$. On the other hand, by taking into account (17), we have

$$\frac{b_2}{a_1}\overline{u} = \overline{t} \in \left(\frac{b_2}{a_1}\frac{a_1}{1+a_1b_2}, \frac{b_2}{a_1}a_1\right) = \left(\frac{b_2}{1+a_1b_2}, b_2\right)$$

and

$$\frac{b_1}{a_2}\overline{w} = \overline{v} \in \left(\frac{b_1}{a_2}\frac{a_2}{1+a_2b_1}, \frac{b_1}{a_2}a_2\right) = \left(\frac{b_1}{1+a_2b_1}, b_1\right),$$

which completes the proof. ∎

Theorem 2.3. The unique equilibrium $(\overline{u}, \overline{t}, \overline{w}, \overline{v}) = \left(\overline{u}, \frac{b_2}{a_1}\overline{u}, \overline{w}, \frac{b_1}{a_2}\overline{w}\right)$ of the system (7) is locally asymptotically stable.

Proof. We define the maps

$$F: \left(\frac{a_1}{1+a_1b_2}, a_1\right) \times \left(\frac{b_2}{1+a_1b_2}, b_2\right) \to \left(\frac{a_1}{1+a_1b_2}, a_1\right) \times \left(\frac{b_2}{1+a_1b_2}, b_2\right)$$

and

$$G: \left(\frac{a_2}{1+a_2b_1}, a_2\right) \times \left(\frac{b_1}{1+a_2b_1}, b_1\right) \to \left(\frac{a_2}{1+a_2b_1}, a_2\right) \times \left(\frac{b_1}{1+a_2b_1}, b_1\right),$$

given by

$$F\begin{pmatrix} x \\ k \end{pmatrix} = \begin{pmatrix} \frac{a_1}{1+xk} \\ \frac{b_2(1+xk)}{1+xk+a_1k} \end{pmatrix} \text{ and } G\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} \frac{a_2}{1+yz} \\ \frac{b_1(1+yz)}{1+yz+a_2y} \end{pmatrix}.$$

The Jacobian matrices evaluated at $\left(\overline{u}, \frac{b_2}{a_1}\overline{u}\right)$ of F and $\left(\overline{w}, \frac{b_1}{a_2}\overline{w}\right)$ of G are

$$J_F(\overline{u}, \overline{t}) = \begin{pmatrix} \frac{-b_2\overline{u}^3}{a_1^2} & \frac{-\overline{u}^3}{a_1} \\ \frac{b_2^3\overline{u}^6}{a_1^5} & \frac{-b_2\overline{u}^4}{a_1^3} \end{pmatrix}, \quad J_G(\overline{w}, \overline{v}) = \begin{pmatrix} \frac{-b_1\overline{w}^3}{a_2^2} & \frac{-\overline{w}^3}{a_2} \\ \frac{b_1^3\overline{w}^6}{a_2^5} & \frac{-b_1\overline{w}^4}{a_2^3} \end{pmatrix}$$

and theirs characteristic equations associated with $\left(\overline{u}, \frac{b_2}{a_1}\overline{u}\right)$ and $\left(\overline{w}, \frac{b_1}{a_2}\overline{w}\right)$ are

$$\begin{aligned}\lambda^2 + \frac{a_1^4 b_2 \overline{u}^3 + a_1^3 b_2 \overline{u}^4}{a_1^6}\lambda + \frac{a_1 b_2^2 \overline{u}^7 + b_2^3 \overline{u}^9}{a_1^6} &= 0,\\ \widehat{\lambda}^2 + \frac{a_2^4 b_1 \overline{w}^3 + a_2^3 b_1 \overline{w}^4}{a_2^6}\widehat{\lambda} + \frac{a_2 b_1^2 \overline{w}^7 + b_1^3 \overline{w}^9}{a_2^6} &= 0,\end{aligned}$$

respectively. Therefore, from Lemma 1.1-(b), we have the following inequalities

$$\begin{aligned}\left|\frac{a_1^4 b_2 \overline{u}^3 + a_1^3 b_2 \overline{u}^4}{a_1^6}\right| &< 1 + \frac{a_1 b_2^2 \overline{u}^7 + b_2^3 \overline{u}^9}{a_1^6} < 2,\\ \left|\frac{a_2^4 b_1 \overline{w}^3 + a_2^3 b_1 \overline{w}^4}{a_2^6}\right| &< 1 + \frac{a_2 b_1^2 \overline{w}^7 + b_1^3 \overline{w}^9}{a_2^6} < 2.\end{aligned}$$

After some calculations from the last inequalities, we obtain the following inequalities:

$$\left(a_1 - \overline{u}\right)^2 + \overline{u}^2 > 0, \quad 8a_1 b_2 + 1 > 0$$

and

$$\left(a_2 - \overline{w}\right)^2 + \overline{w}^2 > 0, \quad 8a_2 b_1 + 1 > 0,$$

which always hold. So, the proof is completed. ∎

Theorem 2.4. The system (7) has not positive periodic solutions with prime period two.

Proof. First, we suppose that the system (7) has positive periodic solutions with prime period two as follows:

$$\left\{\ldots, \left(\phi_1, \theta_1, \alpha_1, \psi_1\right), \left(\phi_2, \theta_2, \alpha_2, \psi_2\right), \ldots\right\},$$

where $\phi_1 \neq \phi_2$, $\theta_1 \neq \theta_2$, $\alpha_1 \neq \alpha_2$ and $\psi_1 \neq \psi_2$. Then we have

$$\phi_1 = \frac{a_1}{1 + \phi_2 \psi_2}, \quad \phi_2 = \frac{a_1}{1 + \phi_1 \psi_1}, \quad \psi_1 = \frac{b_2\left(1 + \phi_2 \psi_2\right)}{1 + \phi_2 \psi_2 + a_1 \psi_2}, \quad \psi_2 = \frac{b_2\left(1 + \phi_1 \psi_1\right)}{1 + \phi_1 \psi_1 + a_1 \psi_1}, \tag{22}$$

$$\alpha_1 = \frac{a_2}{1 + \alpha_2 \theta_2}, \quad \alpha_2 = \frac{a_2}{1 + \alpha_1 \theta_1}, \quad \theta_1 = \frac{b_1\left(1 + \alpha_2 \theta_2\right)}{1 + \alpha_2 \theta_2 + a_2 \theta_2}, \quad \theta_2 = \frac{b_1\left(1 + \alpha_1 \theta_1\right)}{1 + \alpha_1 \theta_1 + a_2 \theta_1}, \tag{23}$$

from which it follows that

$$\psi_1 = \frac{b_2}{1 + \phi_1 \psi_2}, \quad \psi_2 = \frac{b_2}{1 + \phi_2 \psi_1}, \quad \theta_1 = \frac{b_1}{1 + \alpha_1 \theta_2}, \quad \theta_2 = \frac{b_1}{1 + \alpha_2 \theta_1}. \tag{24}$$

From the first two equations of (22), the first two equations of (23) and (24), we have

$$\phi_1 \phi_2 \left(\psi_2 - \psi_1\right) + \phi_1 - \phi_2 = 0, \quad \psi_1 \psi_2 \left(\phi_1 - \phi_2\right) + \psi_1 - \psi_2 = 0, \tag{25}$$

$$\alpha_1 \alpha_2 \left(\theta_2 - \theta_1\right) + \alpha_1 - \alpha_2 = 0, \quad \theta_1 \theta_2 \left(\alpha_1 - \alpha_2\right) + \theta_1 - \theta_2 = 0. \tag{26}$$

(24) implies $\phi_1 \phi_2 \psi_1 \psi_2 = -1$, $\alpha_1 \alpha_2 \theta_1 \theta_2 = -1$ which is a contradiction. So, the proof is completed. ∎

2.2 Closed form solutions of the system (3)

In this subsection, we obtain a closed form solutions of the system (3). By applying the change of variables

$$x_n = \frac{p_{n-1}}{r_n}, \quad y_n = \frac{r_{n-1}}{p_n}, \; n \geq -1, \tag{27}$$

to the system (3), we have the following third-order linear system

$$r_{n+1} - \frac{1}{\alpha_n} p_n - \frac{1}{\alpha_n} p_{n-2} = 0, \quad p_{n+1} - \frac{1}{\beta_n} r_n - \frac{1}{\beta_n} r_{n-2} = 0, \; n \in \mathbb{N}_0, \tag{28}$$

where $p_0 = 1$, $p_{-1} = x_0$, $p_{-2} = x_{-1}y_0$, $r_0 = 1$, $r_{-1} = y_0$, $r_{-2} = y_{-1}x_0$. From the system (28), we have

$$r_{2n+1} - \frac{1}{a_1} p_{2n} - \frac{1}{a_1} p_{2n-2} = 0, \quad r_{2n+2} - \frac{1}{b_1} p_{2n+1} - \frac{1}{b_1} p_{2n-1} = 0, \; n \in \mathbb{N}_0, \tag{29}$$

$$p_{2n+1} - \frac{1}{a_2} r_{2n} - \frac{1}{a_2} r_{2n-2} = 0, \quad p_{2n+2} - \frac{1}{b_2} r_{2n+1} - \frac{1}{b_2} r_{2n-1} = 0, \; n \in \mathbb{N}_0, \tag{30}$$

from which it follows that

$$\begin{aligned} r_{2n+1} - \frac{1}{a_1 b_2} r_{2n-1} - \frac{2}{a_1 b_2} r_{2n-3} - \frac{1}{a_1 b_2} r_{2n-5} = 0, \\ r_{2n+2} - \frac{1}{a_2 b_1} r_{2n} - \frac{2}{a_2 b_1} r_{2n-2} - \frac{1}{a_2 b_1} r_{2n-4} = 0, \end{aligned} \tag{31}$$

$$\begin{aligned} p_{2n+1} - \frac{1}{a_2 b_1} p_{2n-1} - \frac{2}{a_2 b_1} p_{2n-3} - \frac{1}{a_2 b_1} p_{2n-5} = 0, \\ p_{2n+2} - \frac{1}{a_1 b_2} p_{2n} - \frac{2}{a_1 b_2} p_{2n-2} - \frac{1}{a_1 b_2} p_{2n-4} = 0, \end{aligned} \tag{32}$$

for $n \in \mathbb{N}_0$. Equations (31) and (32) have the characteristic equations as follows:

$$\begin{aligned} P_1(\lambda) &= \lambda^6 - \frac{1}{a_1 b_2} \lambda^4 - \frac{2}{a_1 b_2} \lambda^2 - \frac{1}{a_1 b_2} = 0, \\ P_2(\lambda) &= \lambda^6 - \frac{1}{a_2 b_1} \lambda^4 - \frac{2}{a_2 b_1} \lambda^2 - \frac{1}{a_2 b_1} = 0. \end{aligned} \tag{33}$$

Let

$$Q_1(\lambda) = \lambda^3 - \frac{1}{\sqrt{a_1 b_2}} \lambda^2 - \frac{1}{\sqrt{a_1 b_2}}, \quad R_1(\lambda) = \lambda^3 + \frac{1}{\sqrt{a_1 b_2}} \lambda^2 + \frac{1}{\sqrt{a_1 b_2}}, \tag{34}$$

$$Q_2(\lambda) = \lambda^3 - \frac{1}{\sqrt{a_2 b_1}} \lambda^2 - \frac{1}{\sqrt{a_2 b_1}}, \quad R_2(\lambda) = \lambda^3 + \frac{1}{\sqrt{a_2 b_1}} \lambda^2 + \frac{1}{\sqrt{a_2 b_1}}. \tag{35}$$

Then $P_1(\lambda) = Q_1(\lambda) R_1(\lambda)$ and $P_2(\lambda) = Q_2(\lambda) R_2(\lambda)$. Note that the polynomials Q_1, R_1, Q_2 and R_2 satisfy the relations $Q_1(-\lambda) = -R_1(\lambda)$ and $Q_2(-\lambda) = -R_2(\lambda)$. Namely, if λ is any zero of the polynomial R_1, then $-\lambda$ is a zero of the polynomial Q_1 and if λ is any zero of the polynomial

R_2, then $-\lambda$ is a zero of the polynomial Q_2. On the other hand, we consider the following linear equations

$$s_n - \frac{1}{a_1b_2}s_{n-1} - \frac{2}{a_1b_2}s_{n-2} - \frac{1}{a_1b_2}s_{n-3} = 0, \quad \widehat{s}_n - \frac{1}{a_2b_1}\widehat{s}_{n-1} - \frac{2}{a_2b_1}\widehat{s}_{n-2} - \frac{1}{a_2b_1}\widehat{s}_{n-3} = 0. \quad (36)$$

Characteristic equations of equations in (36) are

$$P_1\left(\sqrt{\mu}\right) = \mu^3 - \frac{1}{a_1b_2}\mu^2 - \frac{2}{a_1b_2}\mu - \frac{1}{a_1b_2} = 0$$

and

$$P_2\left(\sqrt{\widehat{\mu}}\right) = \widehat{\mu}^3 - \frac{1}{a_2b_1}\widehat{\mu}^2 - \frac{2}{a_2b_1}\widehat{\mu} - \frac{1}{a_2b_1} = 0.$$

We see from Lemma 1.2 that the equations $P_1\left(\sqrt{\mu}\right) = 0$ and $P_2\left(\sqrt{\widehat{\mu}}\right) = 0$ have one real root and two complex roots denoted by $\widetilde{p}^2$, $\widetilde{r}e^{\pm 2i\theta}$, $\theta \in (0, \pi)$ and $\widehat{p}^2$, $\widehat{r}e^{\pm 2i\theta}$, $\theta \in (0, \pi)$, respectively. These notations are legal, since $\mu = \lambda^2$ and $\widehat{\mu} = \lambda^2$. Also, note that since $a_1b_2 > 0$, $a_2b_1 > 0$ and $\mu^3 = \frac{1}{a_1b_2}(\mu+1)^2$, $\widehat{\mu}^3 = \frac{1}{a_2b_1}(\widehat{\mu}+1)^2$, the unique real roots of $P_1\left(\sqrt{\mu}\right) = 0$ and $P_2\left(\sqrt{\widehat{\mu}}\right) = 0$ are positive. So, we have the general solutions of (36) as follows:

$$s_{n-1} = C_1\widetilde{p}^{2n} + \widetilde{r}^{2n}\left(C_2\cos 2n\theta + C_3\sin 2n\theta\right), \quad n \geq -1, \quad (37)$$

where C_1, C_2, and C_3 are arbitrary constants.

$$\widehat{s}_{n-1} = \widehat{C}_1\widehat{p}^{2n} + \widehat{r}^{2n}\left(\widehat{C}_2\cos 2n\theta + \widehat{C}_3\sin 2n\theta\right), \quad n \geq -1, \quad (38)$$

where $\widehat{C}_1$, $\widehat{C}_2$, and $\widehat{C}_3$ are arbitrary constants. Any solutions of the equations in (31) and (32) are the solutions of the equations in (36). Therefore, we can formulate the sequences $(p_{2n})_{n\geq -1}$ and $(r_{2n})_{n\geq -1}$ as follows:

$$p_{2n} = C_1\widetilde{p}^{2n} + \widetilde{r}^{2n}\left(C_2\cos 2n\theta + C_3\sin 2n\theta\right), \quad n \geq -1, \quad (39)$$

where

$$C_1 = \frac{\widetilde{p}^4\left[1 + \widetilde{r}^4\left(a_1 - x_{-1}\right)y_0 - 2\widetilde{r}^2\cos 2\theta x_{-1}y_0\right]}{\widetilde{p}^4 + \widetilde{r}^4 - 2\widetilde{p}^2\widetilde{r}^2\cos 2\theta},$$

$$C_2 = \frac{\widetilde{r}^2\left[2\widetilde{p}^2\cos 2\theta\left(-1 + \widetilde{p}^2x_{-1}y_0\right) + \widetilde{r}^2\left(1 + \widetilde{p}^4\left(-a_1 + x_{-1}\right)y_0\right)\right]}{\widetilde{p}^4 + \widetilde{r}^4 - 2\widetilde{p}^2\widetilde{r}^2\cos 2\theta},$$

$$C_3 = \frac{\widetilde{r}^2\csc 2\theta\left[\widetilde{r}^4\left(\widetilde{p}^2\left(a_1 - x_{-1}\right) - x_{-1}\right)y_0 - \widetilde{r}^2\cos 2\theta\left(-1 + \widetilde{p}^4\left(a_1 - x_{-1}\right)y_0\right) + \widetilde{p}^2\cos 4\theta\left(-1 + \widetilde{p}^2x_{-1}y_0\right)\right]}{\widetilde{p}^4 + \widetilde{r}^4 - 2\widetilde{p}^2\widetilde{r}^2\cos 2\theta},$$

$$r_{2n} = \widehat{C}_1\widehat{p}^{2n} + \widehat{r}^{2n}\left(\widehat{C}_2\cos 2n\theta + \widehat{C}_3\sin 2n\theta\right), \quad n \geq -1, \quad (40)$$

where

$$\widehat{C}_1 = \frac{\widehat{p}^4\left[1 + \widehat{r}^4\left(a_2 - y_{-1}\right)x_0 - 2\widehat{r}^2\cos 2\theta y_{-1}x_0\right]}{\widehat{p}^4 + \widehat{r}^4 - 2\widehat{p}^2\widehat{r}^2\cos 2\theta},$$

$$\widehat{C}_2 = \frac{\widehat{r}^2\left[2\widehat{p}^2\cos 2\theta\left(-1+\widehat{p}^2 y_{-1}x_0\right)+\widehat{r}^2\left(1+\widehat{p}^4\left(-a_2+y_{-1}\right)x_0\right)\right]}{\widehat{p}^4+\widehat{r}^4-2\widehat{p}^2\widehat{r}^2\cos 2\theta},$$

$$\widehat{C}_3 = \frac{\widehat{r}^2\csc 2\theta\left[\widehat{r}^4\left(\widehat{p}^2\left(a_2-y_{-1}\right)-y_{-1}\right)x_0-\widehat{r}^2\cos 2\theta\left(-1+\widehat{p}^4\left(a_2-y_{-1}\right)x_0\right)+\widehat{p}^2\cos 4\theta\left(-1+\widehat{p}^2 y_{-1}x_0\right)\right]}{\widehat{p}^4+\widehat{r}^4-2\widehat{p}^2\widehat{r}^2\cos 2\theta}.$$

On the other hand, by the first equations of (29), (30) and some operations, we have

$$\begin{aligned} r_{2n+1} &= \frac{1}{a_1}p_{2n}+\frac{1}{a_1}p_{2n-2}, \\ &= C_1\frac{\widetilde{p}^2+1}{a_1}\widetilde{p}^{2n-2}+\frac{\widetilde{r}^{2n}}{a_1}\left(C_2'\cos 2n\theta+C_3'\sin 2n\theta\right), \quad n\geq -1, \end{aligned} \tag{41}$$

where

$$C_2' = C_2+\frac{C_2\cos 2\theta-C_3\sin 2\theta}{\widetilde{r}^2}, \quad C_3' = C_3+\frac{C_3\cos 2\theta+C_2\sin 2\theta}{\widetilde{r}^2},$$

$$\begin{aligned} p_{2n+1} &= \frac{1}{a_2}r_{2n}+\frac{1}{a_2}r_{2n-2}, \\ &= \widehat{C}_1\frac{\widehat{p}^2+1}{a_2}\widehat{p}^{2n-2}+\frac{\widehat{r}^{2n}}{a_2}\left(\widehat{C}_2'\cos 2n\theta+\widehat{C}_3'\sin 2n\theta\right), \quad n\geq -1, \end{aligned} \tag{42}$$

where

$$\widehat{C}_2' = \widehat{C}_2+\frac{\widehat{C}_2\cos 2\theta-\widehat{C}_3\sin 2\theta}{\widehat{r}^2}, \quad \widehat{C}_3' = \widehat{C}_3+\frac{\widehat{C}_3\cos 2\theta+\widehat{C}_2\sin 2\theta}{\widehat{r}^2}.$$

Also, the relations $P_1(\lambda)=Q_1(\lambda)R_1(\lambda)$ and $P_2(\lambda)=Q_2(\lambda)R_2(\lambda)$ and $Q_1(-\lambda)=-R_1(\lambda)$ and $Q_2(-\lambda)=-R_2(\lambda)$ imply that $\widetilde{p}$ is the root of $Q_1(\lambda)$ and $-\widetilde{p}$ is the root of $R_1(\lambda)$, $\widehat{p}$ is the root of $Q_2(\lambda)$ and $-\widehat{p}$ is the root of $R_2(\lambda)$. Hence $\widetilde{p}$ and $\widehat{p}$ satisfy the following relations:

$$\frac{\widetilde{p}^2+1}{a_1} = \sqrt{\frac{b_2}{a_1}}\widetilde{p}^3, \quad \frac{\widehat{p}^2+1}{a_2} = \sqrt{\frac{b_1}{a_2}}\widehat{p}^3.$$

From these and (41), (42) follows that

$$r_{2n+1} = C_1\sqrt{\frac{b_2}{a_1}}\widetilde{p}^{2n+1}+\frac{\widetilde{r}^{2n}}{a_1}\left(C_2'\cos 2n\theta+C_3'\sin 2n\theta\right), \quad n\geq -1, \tag{43}$$

where

$$C_2' = C_2+\frac{C_2\cos 2\theta-C_3\sin 2\theta}{\widetilde{r}^2}, \quad C_3' = C_3+\frac{C_3\cos 2\theta+C_2\sin 2\theta}{\widetilde{r}^2},$$

$$p_{2n+1} = \widehat{C}_1\sqrt{\frac{b_1}{a_2}}\widehat{p}^{2n+1}+\frac{\widehat{r}^{2n}}{a_2}\left(\widehat{C}_2'\cos 2n\theta+\widehat{C}_3'\sin 2n\theta\right), \quad n\geq -1, \tag{44}$$

where

$$\widehat{C}_2' = \widehat{C}_2+\frac{\widehat{C}_2\cos 2\theta-\widehat{C}_3\sin 2\theta}{\widehat{r}^2}, \quad \widehat{C}_3' = \widehat{C}_3+\frac{\widehat{C}_3\cos 2\theta+\widehat{C}_2\sin 2\theta}{\widehat{r}^2}.$$

Therefore, from (27), (39), (40), (43), (44), we have the closed form solutions of the system (3) as follows:

$$x_{2n} = \frac{\widehat{C}_1\sqrt{\frac{b_1}{a_2}}\widehat{p}^{2n-1}+\frac{\widehat{r}^{2n-2}}{a_2}\left(\widehat{C}_2'\cos(2n-2)\theta+\widehat{C}_3'\sin(2n-2)\theta\right)}{\widehat{C}_1\widehat{p}^{2n}+\widehat{r}^{2n}\left(\widehat{C}_2\cos 2n\theta+\widehat{C}_3\sin 2n\theta\right)}, \tag{45}$$

$$x_{2n+1}=\frac{C_1\widetilde{p}^{2n}+\widetilde{r}^{2n}\left(C_2\cos 2n\theta+C_3\sin 2n\theta\right)}{C_1\sqrt{\frac{b_2}{a_1}}\widetilde{p}^{2n+1}+\frac{\widetilde{r}^{2n}}{a_1}\left(C_2'\cos 2n\theta+C_3'\sin 2n\theta\right)}, \tag{46}$$

$$y_{2n}=\frac{C_1\sqrt{\frac{b_2}{a_1}}\widetilde{p}^{2n-1}+\frac{\widetilde{r}^{2n-2}}{a_1}\left(C_2'\cos\left(2n-2\right)\theta+C_3'\sin\left(2n-2\right)\theta\right)}{C_1\widetilde{p}^{2n}+\widetilde{r}^{2n}\left(C_2\cos 2n\theta+C_3\sin 2n\theta\right)} \tag{47}$$

and

$$y_{2n+1}=\frac{\widehat{C}_1\widehat{p}^{2n}+\widehat{r}^{2n}\left(\widehat{C}_2\cos 2n\theta+\widehat{C}_3\sin 2n\theta\right)}{\widehat{C}_1\sqrt{\frac{b_1}{a_2}}\widehat{p}^{2n+1}+\frac{\widehat{r}^{2n}}{a_2}\left(\widehat{C}_2'\cos 2n\theta+\widehat{C}_3'\sin 2n\theta\right)}, \tag{48}$$

where

$$C_1=\frac{\widetilde{p}^4\left[1+\widetilde{r}^4\left(a_1-x_{-1}\right)y_0-2\widetilde{r}^2\cos 2\theta x_{-1}y_0\right]}{\widetilde{p}^4+\widetilde{r}^4-2\widetilde{p}^2\widetilde{r}^2\cos 2\theta},$$

$$C_2=\frac{\widetilde{r}^2\left[2\widetilde{p}^2\cos 2\theta\left(-1+\widetilde{p}^2x_{-1}y_0\right)+\widetilde{r}^2\left(1+\widetilde{p}^4\left(-a_1+x_{-1}\right)y_0\right)\right]}{\widetilde{p}^4+\widetilde{r}^4-2\widetilde{p}^2\widetilde{r}^2\cos 2\theta},$$

$$C_3=\frac{\widetilde{r}^2\csc 2\theta\left[\widetilde{r}^4\left(\widetilde{p}^2\left(a_1-x_{-1}\right)-x_{-1}\right)y_0-\widetilde{r}^2\cos 2\theta\left(-1+\widetilde{p}^4\left(a_1-x_{-1}\right)y_0\right)+\widetilde{p}^2\cos 4\theta\left(-1+\widetilde{p}^2x_{-1}y_0\right)\right]}{\widetilde{p}^4+\widetilde{r}^4-2\widetilde{p}^2\widetilde{r}^2\cos 2\theta},$$

$$C_2'=C_2+\frac{C_2\cos 2\theta-C_3\sin 2\theta}{\widetilde{r}^2},\quad C_3'=C_3+\frac{C_3\cos 2\theta+C_2\sin 2\theta}{\widetilde{r}^2},$$

$$\widehat{C}_1=\frac{\widehat{p}^4\left[1+\widehat{r}^4\left(a_2-y_{-1}\right)x_0-2\widehat{r}^2\cos 2\theta y_{-1}x_0\right]}{\widehat{p}^4+\widehat{r}^4-2\widehat{p}^2\widehat{r}^2\cos 2\theta},$$

$$\widehat{C}_2=\frac{\widehat{r}^2\left[2\widehat{p}^2\cos 2\theta\left(-1+\widehat{p}^2y_{-1}x_0\right)+\widehat{r}^2\left(1+\widehat{p}^4\left(-a_2+y_{-1}\right)x_0\right)\right]}{\widehat{p}^4+\widehat{r}^4-2\widehat{p}^2\widehat{r}^2\cos 2\theta},$$

$$\widehat{C}_3=\frac{\widehat{r}^2\csc 2\theta\left[\widehat{r}^4\left(\widehat{p}^2\left(a_2-y_{-1}\right)-y_{-1}\right)x_0-\widehat{r}^2\cos 2\theta\left(-1+\widehat{p}^4\left(a_2-y_{-1}\right)x_0\right)+\widehat{p}^2\cos 4\theta\left(-1+\widehat{p}^2y_{-1}x_0\right)\right]}{\widehat{p}^4+\widehat{r}^4-2\widehat{p}^2\widehat{r}^2\cos 2\theta},$$

$$\widehat{C}_2'=\widehat{C}_2+\frac{\widehat{C}_2\cos 2\theta-\widehat{C}_3\sin 2\theta}{\widehat{r}^2},\quad \widehat{C}_3'=\widehat{C}_3+\frac{\widehat{C}_3\cos 2\theta+\widehat{C}_2\sin 2\theta}{\widehat{r}^2}.$$

2.3 Globally asymptotically stability

In this subsection, we study globally asymptotically stability of the unique positive equilibrium $\left(\overline{u},\overline{t}\right)=\left(\overline{u},\frac{b_2}{a_1}\overline{u}\right)$, $\left(\overline{w},\overline{v}\right)=\left(\overline{w},\frac{b_1}{a_2}\overline{w}\right)$ of the system (7).

Lemma 2.5. Consider the cubic polynomial $S\left(\lambda\right)=\lambda^3-c\lambda^2-c$, where c is a real number. Then zeros of the polynomial S satisfy the relation $|\sigma|<\rho$, where ρ is the unique real zero of the polynomial S and σ is one of complex conjugate ones.

Proof. Note that $c=\rho\sigma\bar{\sigma}=\rho\left|\sigma\right|^2$. Since $S\left(\rho\right)=0$, we have

$$\rho^3-c\rho^2-c=\rho^3-\rho\left|\sigma\right|^2\rho^2-\rho\left|\sigma\right|^2=0$$

which implies

$$\left|\sigma\right|^2=\frac{\rho^2}{\rho^2+1}<\rho^2.$$

Therefore, the proof is completed. ∎

Theorem 2.6. The unique equilibrium $\left(\overline{u},\overline{t}\right)=\left(\overline{u},\frac{b_2}{a_1}\overline{u}\right)$, $\left(\overline{w},\overline{v}\right)=\left(\overline{w},\frac{b_1}{a_2}\overline{w}\right)$ of the system (7) is globally asymptotically stable.

Proof. We know from Theorem 2.3 that the unique equilibrium $\left(\overline{u},\overline{t}\right)=\left(\overline{u},\frac{b_2}{a_1}\overline{u}\right)$, $\left(\overline{w},\overline{v}\right)=\left(\overline{w},\frac{b_1}{a_2}\overline{w}\right)$ of the system (7) is locally asymptotically stable. Hence, it is enough to show that

$$\lim_{n\to\infty}u_n=\overline{u},\quad \lim_{n\to\infty}t_n=\overline{t},\quad \lim_{n\to\infty}w_n=\overline{w}\ \text{ and }\ \lim_{n\to\infty}v_n=\overline{v},$$

or

$$\lim_{n\to\infty}x_{2n}=\overline{v},\quad \lim_{n\to\infty}x_{2n+1}=\overline{u},\quad \lim_{n\to\infty}y_{2n}=\overline{t}\ \text{ and }\ \lim_{n\to\infty}y_{2n+1}=\overline{w},$$

by taking into account (6). We also know that $\overline{u}$ and $\overline{w}$ are the unique real zeros of the polynomials P and R in (18). On the other hand, $\widetilde{p}$ is the unique real zero of polynomial Q_1 in (34) and $\widehat{p}$ is the unique real zero of polynomial Q_2 in (35). We claim that the zeros of the polynomials P and Q_1 and also the zeros of the polynomials R and Q_2 are of the relations

$$\sqrt{\frac{a_1}{b_2}}\frac{1}{\widetilde{p}}=\overline{u},\quad \sqrt{\frac{a_2}{b_1}}\frac{1}{\widehat{p}}=\overline{w}, \tag{49}$$

respectively. To verify these relations, we have

$$\begin{aligned}
P\left(\overline{u}\right) &= \overline{u}^3+\frac{a_1}{b_2}\overline{u}-\frac{a_1^2}{b_2}\\
&= \left(\sqrt{\frac{a_1}{b_2}}\frac{1}{\widetilde{p}}\right)^3+\frac{a_1}{b_2}\sqrt{\frac{a_1}{b_2}}\frac{1}{\widetilde{p}}-\frac{a_1^2}{b_2}\\
&= -\left(\frac{a_1^2}{b_2}\frac{1}{\widetilde{p}^3}\right)\left(\widetilde{p}^3-\frac{1}{\sqrt{a_1b_2}}\widetilde{p}^2-\frac{1}{\sqrt{a_1b_2}}\right)\\
&= -\left(\frac{a_1^2}{b_2}\frac{1}{\widetilde{p}^3}\right)Q_1\left(\widetilde{p}\right)\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
R\left(\overline{w}\right) &= \overline{w}^3+\frac{a_2}{b_1}\overline{w}-\frac{a_2^2}{b_1}\\
&= \left(\sqrt{\frac{a_2}{b_1}}\frac{1}{\widehat{p}}\right)^3+\frac{a_2}{b_1}\sqrt{\frac{a_2}{b_1}}\frac{1}{\widehat{p}}-\frac{a_2^2}{b_1}\\
&= -\left(\frac{a_2^2}{b_1}\frac{1}{\widehat{p}^3}\right)\left(\widehat{p}^3-\frac{1}{\sqrt{a_2b_1}}\widehat{p}^2-\frac{1}{\sqrt{a_2b_1}}\right)\\
&= -\left(\frac{a_2^2}{b_1}\frac{1}{\widehat{p}^3}\right)Q_2\left(\widehat{p}\right)\\
&= 0.
\end{aligned}$$

By taking limits of (45)-(48) as $n \to \infty$ by using (49) and the result of Lemma 2.5, we have

$$\begin{aligned}
\lim_{n\to\infty} x_{2n} &= \lim_{n\to\infty} \frac{\widehat{p}^{2n-1}}{\widehat{p}^{2n}} \frac{\widehat{C}_1\sqrt{\frac{b_1}{a_2}} + \left(\frac{\widehat{r}}{\widehat{p}}\right)^{2n-1} \frac{1}{a_2\widehat{r}} \left(\widehat{C}_2' \cos\left(2n-2\right)\theta + \widehat{C}_3' \sin\left(2n-2\right)\theta\right)}{\widehat{C}_1 + \left(\frac{\widehat{r}}{\widehat{p}}\right)^{2n} \left(\widehat{C}_2 \cos 2n\theta + \widehat{C}_3 \sin 2n\theta\right)} \\
&= \sqrt{\frac{b_1}{a_2}} \frac{1}{\widehat{p}} \\
&= \frac{b_1}{a_2} \overline{w} \\
&= \overline{v}
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} x_{2n+1} &= \lim_{n\to\infty} \frac{\widetilde{p}^{2n}}{\widetilde{p}^{2n+1}} \frac{C_1 + \left(\frac{\widetilde{r}}{\widetilde{p}}\right)^{2n} \left(C_2 \cos 2n\theta + C_3 \sin 2n\theta\right)}{C_1\sqrt{\frac{b_2}{a_1}} + \left(\frac{\widetilde{r}}{\widetilde{p}}\right)^{2n+1} \frac{1}{a_1\widetilde{r}} + \left(C_2' \cos 2n\theta + C_3' \sin 2n\theta\right)} \\
&= \sqrt{\frac{a_1}{b_2}} \frac{1}{\widetilde{p}} \\
&= \overline{u}
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} y_{2n} &= \lim_{n\to\infty} \frac{\widetilde{p}^{2n-1}}{\widetilde{p}^{2n}} \frac{C_1\sqrt{\frac{b_2}{a_1}} + \left(\frac{\widetilde{r}}{\widetilde{p}}\right)^{2n-1} \frac{1}{a_1\widetilde{r}} \left(C_2' cos\left(2n-2\right)\theta + C_3' sin\left(2n-2\right)\theta\right)}{C_1 + \left(\frac{\widetilde{r}}{\widetilde{p}}\right)^{2n} \left(C_2 cos 2n\theta + C_3 sin 2n\theta\right)} \\
&= \sqrt{\frac{b_2}{a_1}} \frac{1}{\widetilde{p}} \\
&= \frac{b_2}{a_1} \overline{u} \\
&= \overline{t}
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} y_{2n+1} &= \lim_{n\to\infty} \frac{\widehat{p}^{2n}}{\widehat{p}^{2n+1}} \frac{\widehat{C}_1 + \left(\frac{\widehat{r}}{\widehat{p}}\right)^{2n} \left(\widehat{C}_2 \cos 2n\theta + \widehat{C}_3 \sin 2n\theta\right)}{\widehat{C}_1\sqrt{\frac{b_1}{a_2}} + \left(\frac{\widehat{r}}{\widehat{p}}\right)^{2n+1} \frac{1}{a_2\widehat{r}} + \left(\widehat{C}_2' \cos 2n\theta + \widehat{C}_3' \sin 2n\theta\right)} \\
&= \sqrt{\frac{a_2}{b_1}} \frac{1}{\widehat{p}} \\
&= \overline{w}.
\end{aligned}$$

So, the proof is completed. ∎

Theorem 2.7. The system (3) has positive periodic solutions with prime period two which is given by

$$\left\{\ldots,\left(\overline{u},\frac{b_2}{a_1}\overline{u}\right),\left(\overline{w},\frac{b_1}{a_2}\overline{w}\right),\left(\overline{u},\frac{b_2}{a_1}\overline{u}\right),\left(\overline{w},\frac{b_1}{a_2}\overline{w}\right),\ldots\right\}. \tag{50}$$

Proof. First, we suppose that the system (3) has positive periodic solutions with prime period two as follows:

$$\{\ldots,(\phi,\theta),(\alpha,\psi),(\phi,\theta),(\alpha,\psi),\ldots\}, \tag{51}$$

where $\phi\neq\alpha$ and $\theta\neq\psi$. From (4) and (5), we have

$$\phi=\frac{a_1}{1+\phi\psi},\quad \psi=\frac{b_2}{1+\phi\psi},\quad \theta=\frac{a_2}{1+\alpha\theta},\quad \alpha=\frac{b_1}{1+\alpha\theta}, \tag{52}$$

from which it follows that

$$\psi=\frac{b_2}{a_1}\phi,\quad \alpha=\frac{b_1}{a_2}\theta, \tag{53}$$

By using (52) and (53), we have

$$P(\phi)=\phi^3+\frac{a_1}{b_2}\phi-\frac{a_1^2}{b_2}=0,\quad R(\theta)=\theta^3+\frac{a_2}{b_1}\theta-\frac{a_2^2}{b_1}=0.$$

We know from Lemma 2.2 that each of the last equations has the unique real root such that $\phi=\overline{u}$ and $\theta=\overline{w}$, respectively. Hence, the result follows by (53). ∎

The following corollary is a straightforward result of Theorem 2.6.

Corollary 2.8. Every positive solution of the system (3) tends to its periodic solution with prime period two which is given by (50).

We give the following numerical example to support our theoretical results.

Example 2.9. In the following Figures, we illustrate the solutions of the systems in (3) and (7) which corresponds to the values of initial conditions $x_{-1}=u_0=3.1$, $x_0=v_0=2.3$, $y_{-1}=w_0=5$, $y_0=t_0=3.4$ and to the values of parameters $a_1=13$, $b_1=5$,$a_2=7$, $b_2=3$.

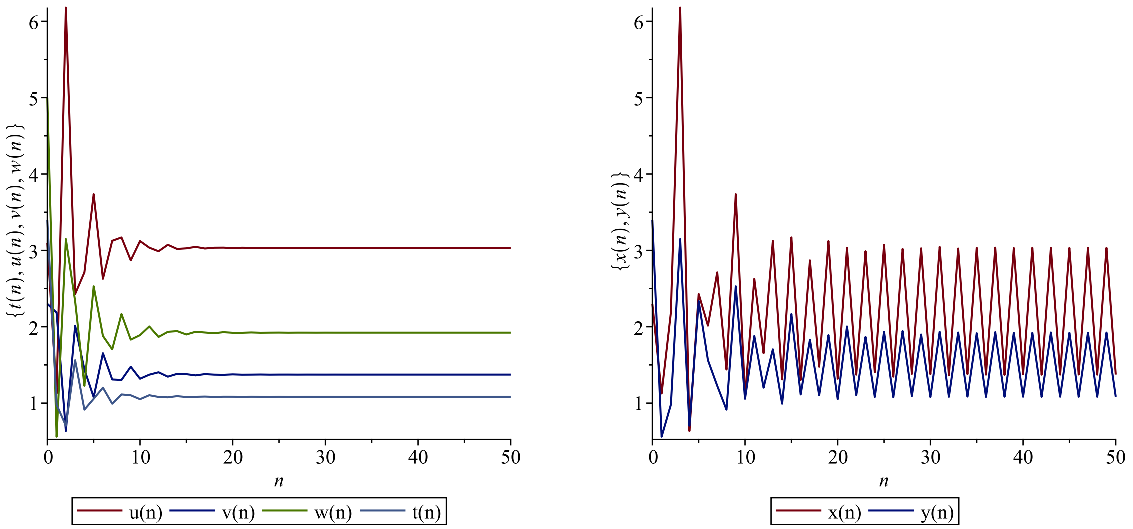

FIGURE 1. $a_1 = 13$, $b_1 = 5$,$a_2 = 7$, $b_2 = 3$.

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