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ON A SOLVABLE SYSTEM OF DIFFERENCE EQUATIONS OF HIGHER-ORDER WITH PERIOD TWO COEFFICIENTS

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Abstract. We show that the next difference equations system

$$x_{n+1}=\frac{a_n x_{n-k+1}y_{n-k}}{y_n-\alpha_n}+\beta_{n+1},\ y_{n+1}=\frac{b_n y_{n-k+1}x_{n-k}}{x_n-\beta_n}+\alpha_{n+1},\ n\in\mathbb{N}_0,$$

where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, the sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$ are two periodic and the initial conditions x_{-i}, y_{-i} $i\in\{0,1,\ldots,k\}$, are non-zero real numbers, can be solved. Also, we investigate the behavior of solutions of above mentioned system when $a_0=b_1$ and $a_1=b_0$.

1. Introduction

Theory of difference equations have attracted attention of many authors in recent years(see, e.g., [1]-[40]). Most of the recent applications of this theory have appeared in many scientific areas such as biology, physics, engineering, economics. Particularly, rational difference equations and their systems of higher order have great importance in applications. It is very worthy to find systems which belong to solvable nonlinear difference equations systems and to solve nonlinear difference equations or systems in closed-form or explicit-form. The found formulas for the solutions of these types of equations or systems can be used easily for description of many features of the solutions of these equations or systems. For this reason, finding of a formula for solution of a nonlinear difference equation is worthy as well as interesting.

In an earlier paper, Elabbasy et al., in [3], considered, among other things, the next difference equation

$$x_{n+1}=\frac{x_n x_{n-1}}{x_n-1}+1,\ n\in\mathbb{N}_0. \tag{1}$$

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Quite recently in [14], Haddad et al. considered the following system of difference equations

$$x_{n+1}=\frac{ax_ny_{n-1}}{y_n-\alpha}+\beta,\ y_{n+1}=\frac{by_nx_{n-1}}{x_n-\beta}+\alpha,\ n\in\mathbb{N}_0, \tag{2}$$

where the parameters a,b,α,β and the initial conditions $x_{-i},y_{-i},i=0,1$, are non-zero real numbers, which is an extension of the equation in (1). By using appropriate substitutions on variables, authors reduced system (2) to the first order linear difference equations and investigated the existence and behavior of the solutions of system (2).

Our aim in this study to show that the next difference equations system

$$x_{n+1}=\frac{a_nx_{n-k+1}y_{n-k}}{y_n-\alpha_n}+\beta_{n+1},\ y_{n+1}=\frac{b_ny_{n-k+1}x_{n-k}}{x_n-\beta_n}+\alpha_{n+1},\ n\in\mathbb{N}_0, \tag{3}$$

where the sequences a_n, b_n, α_n, β_n are two periodic and the initial conditions x_{-i}, y_{-i}, $i\in\{0,1,\ldots,k\}$, are non-zero real numbers, is solvable in closed form. Also, by using obtained formulas we give the behavior and periodicity of well-defined solutions of system (3) when $a_0=b_1$ and $a_1=b_0$. Note that system (3) is a natural extension of both Eq. (1) and system (2),

Lemma 1. [27] *Let $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ be two sequences of real numbers and the sequences y_{km+i}, $i=\overline{0,k-1}$, be solutions of the equations*

$$y_{km+i}=a_{km+i}y_{k(m-1)+i}+b_{km+i},\ m\in\mathbb{N}_0. \tag{4}$$

Then, for each fixed $i\in\{0,1,\ldots,k-1\}$ and $m\geq-1$, equation (4) has the general solution

$$y_{km+i}=y_{-k+i}\prod_{j=0}^{m}a_{kj+i}+\sum_{s=0}^{m}b_{ks+i}\prod_{j=s+1}^{m}a_{kj+i}.$$

Further, if $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are constant and $i=\overline{0,k-1}$, then

$$y_{km+i}=\begin{cases}a^{m+1}y_{-k+i}+b\frac{1-a^{m+1}}{1-a}, & \text{if } a\neq1,\\ y_{-k+i}+b\left(m+1\right), & \text{if } a=1.\end{cases}$$

Definition 2. [12] *(Periodicity) A solution $\{x_n\}_{n=-k}^{\infty}$ of equation $x_{n+1}=f\left(x_n,x_{n-1},\ldots,x_{n-k}\right)$, $n\in\mathbb{N}_0$, is called periodic with period p if there exists an integer $p\geq1$ such that $x_{n+p}=x_n$ for all $n\geq-k$.*

In the sequel, as usual, we suppose that $\prod_{j=i}^{m}A_j=1$ and $\sum_{j=i}^{m}A_j=0$, for all $m<i$.

2. Solutions of System (3)

Let $(x_n,y_n)_{n\geq-k}$ be a solution of system (3). If one of the initial conditions x_{-j}, y_{-j}, $j=\overline{0,k}$ is equal to zero, then the system (3) is not defined. For example, if $x_{-k}=0$, then $y_1=\alpha_1$, and so x_2 can not be calculated. For the other $j=\overline{0,k-1}$,

the case is same. Now assume that $x_{-j} \neq 0 \neq y_{-j}$, $j = \overline{0,k}$. If one of the terms x_{n_0} and y_{n_0}, for $n_0 \geq 1$, is equal to zero, then from system (3) either $x_{n_0+k} = \frac{a_{n_0+k-1}x_{n_0}y_{n_0-1}}{y_{n_0+k-1}-\alpha_{n_0+k-1}} + \beta_{n_0+k} = \beta_{n_0+k}$ and so, it follows that, y_{n_0+k+1} is not defined or $y_{n_0+k} = \frac{b_{n_0+k-1}y_{n_0}x_{n_0-1}}{x_{n_0+k-1}-\beta_{n_0+k-1}} + \alpha_{n_0+k} = \alpha_{n_0+k}$ and so, it follows that, x_{n_0+k+1} is not defined. Thus, for well-defined solution of system (3), we may assume that none of the terms of x_n and y_n is not equal to zero, for every $n \geq -k$. By means of the change of variables

$$u_n = \frac{x_n - \beta_n}{x_{n-k}},\ v_n = \frac{y_n - \alpha_n}{y_{n-k}},\ n \in \mathbb{N}_0, \tag{5}$$

the system in (3) becomes

$$u_{n+1} = \frac{a_n}{v_n},\ v_{n+1} = \frac{b_n}{u_n},\ n \in \mathbb{N}_0. \tag{6}$$

From (6), we have two independent equations

$$u_{n+2} = \frac{a_{n+1}}{b_n}u_n,\ v_{n+2} = \frac{b_{n+1}}{a_n}v_n,\ n \in \mathbb{N}_0, \tag{7}$$

and so

$$u_{2m+i} = \left(\frac{a_{1-i}}{b_i}\right)^m u_i,\ v_{2m+i} = \left(\frac{b_{1-i}}{a_i}\right)^m v_i,\ i \in \{0,1\},\ m \in \mathbb{N}_0, \tag{8}$$

where the sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are two periodic. From (5) we have that

$$\begin{aligned} x_n &= u_n x_{n-k} + \beta_n = u_n u_{n-k} x_{n-2k} + u_n \beta_{n-k} + \beta_n, \\ y_n &= v_n y_{n-k} + \alpha_n = v_n v_{n-k} y_{n-2k} + v_n \alpha_{n-k} + \alpha_n,\ n \in \mathbb{N}_0. \end{aligned} \tag{9}$$

We consider three cases: (a) $k = 1$; (b) $k = 2t+1$ $(t = 1, 2, \dots)$; and (c) $k = (2t)$ $(t = 1, 2, \dots)$.

a)If $k = 1$, then, from equations in (9), we can write

$$x_n = u_n x_{n-1} + \beta_n,\ y_n = v_n y_{n-1} + \alpha_n,\ n \in \mathbb{N}_0. \tag{10}$$

from which it follows that

$$\begin{aligned} x_{2n} &= u_{2n}x_{2n-1} + \beta_{2n} = \left(\frac{a_1}{b_0}\right)^n u_0 x_{2n-1} + \beta_0,\quad n \in \mathbb{N}_0, \\ x_{2n+1} &= u_{2n+1}x_{2n} + \beta_{2n+1} = \left(\frac{a_0}{b_1}\right)^n u_1 x_{2n} + \beta_1,\ n \in \mathbb{N}_0, \\ y_{2n} &= v_{2n}y_{2n-1} + \alpha_{2n} = \left(\frac{b_1}{a_0}\right)^n v_0 y_{2n-1} + \alpha_0,\ n \in \mathbb{N}_0, \\ y_{2n+1} &= v_{2n+1}y_{2n} + \alpha_{2n+1} = \left(\frac{b_0}{a_1}\right)^n v_1 y_{2n} + \alpha_1,\ n \in \mathbb{N}_0. \end{aligned}$$

which implies that

$$x_{2n+1} = \left(\frac{a_0}{b_1}\right)^n \left(\frac{a_1}{b_0}\right)^n u_0u_1x_{2n-1} + \left(\frac{a_0}{b_1}\right)^n u_1\beta_0 + \beta_1,\ n \in \mathbb{N}_0, \quad (11)$$

$$x_{2n+2} = \left(\frac{a_0}{b_1}\right)^n \left(\frac{a_1}{b_0}\right)^{n+1} u_0u_1x_{2n} + \left(\frac{a_1}{b_0}\right)^{n+1} u_0\beta_1 + \beta_0,\ n \in \mathbb{N}_0, \quad (12)$$

$$y_{2n+1} = \left(\frac{b_0}{a_1}\right)^n \left(\frac{b_1}{a_0}\right)^n v_0v_1y_{2n-1} + \left(\frac{b_0}{a_1}\right)^n v_1\alpha_0 + \alpha_1,\ n \in \mathbb{N}_0, \quad (13)$$

$$y_{2n+2} = \left(\frac{b_0}{a_1}\right)^n \left(\frac{b_1}{a_0}\right)^{n+1} v_0v_1y_{2n} + \left(\frac{b_1}{a_0}\right)^{n+1} v_0\alpha_1 + \alpha_0,\ n \in \mathbb{N}_0. \quad (14)$$

Hence, from Lemma 1 for $k = 1$ and Eqs. (11)-(14), we can write the solution of system (3) as follows:

$$\begin{aligned} x_{2n-1} &= x_{-1}\prod_{t=0}^{n-1} d\left(\frac{a_0}{b_1}\right)^t \left(\frac{a_1}{b_0}\right)^t + \sum_{r=0}^{n-1} \beta_0u_1 \left(\frac{a_0}{b_1}\right)^r \prod_{t=r+1}^{n-1} d\left(\frac{a_0}{b_1}\right)^t \left(\frac{a_1}{b_0}\right)^t \\ &+ \sum_{r=0}^{n-1} \beta_1 \prod_{t=r+1}^{n-1} d\left(\frac{a_0}{b_1}\right)^t \left(\frac{a_1}{b_0}\right)^t, \\ x_{2n} &= x_0\prod_{t=0}^{n-1} d\left(\frac{a_0}{b_1}\right)^t \left(\frac{a_1}{b_0}\right)^{t+1} + \sum_{r=0}^{n-1} \beta_0 \prod_{t=r+1}^{n-1} d\left(\frac{a_0}{b_1}\right)^t \left(\frac{a_1}{b_0}\right)^{t+1} \\ &+ \sum_{r=0}^{n-1} \beta_1u_0 \left(\frac{a_1}{b_0}\right)^{r+1} \prod_{t=r+1}^{n-1} d\left(\frac{a_0}{b_1}\right)^t \left(\frac{a_1}{b_0}\right)^{t+1}, \end{aligned}$$

$$\begin{aligned} y_{2n-1} &= y_{-1}\prod_{t=0}^{n-1} \frac{a_0b_0}{d}\left(\frac{b_0}{a_1}\right)^t \left(\frac{b_1}{a_0}\right)^t + \sum_{r=0}^{n-1} \alpha_0v_1 \left(\frac{b_0}{a_1}\right)^r \prod_{t=r+1}^{n-1} \frac{a_0b_0}{d}\left(\frac{b_0}{a_1}\right)^t \left(\frac{b_1}{a_0}\right)^t \\ &+ \sum_{r=0}^{n-1} \alpha_1 \prod_{t=r+1}^{n-1} \frac{a_0b_0}{d}\left(\frac{b_0}{a_1}\right)^t \left(\frac{b_1}{a_0}\right)^t, \\ y_{2n} &= y_0\prod_{t=0}^{n-1} \frac{a_0b_0}{d}\left(\frac{b_0}{a_1}\right)^t \left(\frac{b_1}{a_0}\right)^{t+1} + \sum_{r=0}^{n-1} \alpha_0 \prod_{t=r+1}^{n-1} \frac{a_0b_0}{d}\left(\frac{b_0}{a_1}\right)^t \left(\frac{b_1}{a_0}\right)^{t+1} \\ &+ \sum_{r=0}^{n-1} \alpha_1v_0 \left(\frac{b_1}{a_0}\right)^{r+1} \prod_{t=r+1}^{n-1} \frac{a_0b_0}{d}\left(\frac{b_0}{a_1}\right)^t \left(\frac{b_1}{a_0}\right)^{t+1}, \end{aligned}$$

where $u_0 = \frac{x_0-\beta_0}{x_{-k}}$, $u_1 = \frac{x_1-\beta_1}{x_{1-k}}$, $v_0 = \frac{y_0-\alpha_0}{y_{-k}}$, $v_1 = \frac{y_1-\alpha_1}{y_{1-k}}$, $d = u_0u_1$ and $v_0v_1 = \frac{a_0b_0}{d}$.
b) Suppose $k = 2t + 1$ $(t = 1, 2, \dots)$. Iterating the right-hand side of equations in

(9), we can write

$$\begin{aligned} x_{4tm+2m+2j+1} &= x_{2j-4t-1}\prod_{s=0}^{m} u_{2(2ts+s+j)+1}u_{2(2ts+s+j-t)} \\ &+ \sum_{l=0}^{2m+1}\beta_{2(tl+j-t)+l}\prod_{s=l+1}^{2m+1} u_{2(ts+j-t)+s} \qquad (15) \\ x_{4tm+2m+2j+2} &= x_{2j-4t}\prod_{s=0}^{m} u_{2(2ts+s+j+1)}u_{2(2ts+s+j-t)+1} \\ &+ \sum_{l=0}^{2m+1}\beta_{2(tl+j-t)+l+1}\prod_{s=l+1}^{2m+1} u_{2(ts+j-t)+s+1} \qquad (16) \\ y_{4tm+2m+2j+1} &= y_{2j-4t-1}\prod_{s=0}^{m} v_{2(2ts+s+j)+1}v_{2(2ts+s+j-t)} \\ &+ \sum_{l=0}^{2m+1}\alpha_{2(tl+j-t)+l}\prod_{s=l+1}^{2m+1} v_{2(ts+j-t)+s} \qquad (17) \\ y_{4tm+2m+2j+2} &= y_{2j-4t}\prod_{s=0}^{m} v_{2(2ts+s+j+1)}v_{2(2ts+s+j-t)+1} \\ &+ \sum_{l=0}^{2m+1}\alpha_{2(tl+j-t)+l+1}\prod_{s=l+1}^{2m+1} v_{2(ts+j-t)+s+1}, \qquad (18) \end{aligned}$$

where $j \in \{t, t+1, ..., 3t\}$. Using (8) in (15)-(18), we get

$$\begin{aligned} x_{4tm+2m+2j+1} &= x_{2j-4t-1}\prod_{s=0}^{m} d\left(\frac{a_0}{b_1}\right)^{(2t+1)s+j}\left(\frac{a_1}{b_0}\right)^{(2t+1)s+j-t} \\ &+ \sum_{l=0}^{m}\beta_{2(2tl+j+l-t)}u_0^{m-l}u_1^{m-l+1} \\ &\times\left(\frac{a_1}{b_0}\right)^{(t+j+1)(m-l)+(2t+1)\frac{m(m-1)-l(l-1)}{2}}\left(\frac{a_0}{b_1}\right)^{j(m-l+1)+(2t+1)\frac{m(m+1)-l(l-1)}{2}} \\ &+ \sum_{l=0}^{m}\beta_{2(2tl+j+l)+1}u_0^{m-l}u_1^{m-l}\left(\frac{a_1}{b_0}\right)^{(t+j+1)(m-l)+(2t+1)\frac{m(m-1)-l(l+1)}{2}} \\ &\times\left(\frac{a_0}{b_1}\right)^{j(m-l)+(2t+1)\frac{m(m+1)-l(l+1)}{2}} \end{aligned}$$

$$x_{4tm+2m+2j+2} = x_{2j-4t}\prod_{s=0}^{m} d\left(\frac{a_0}{b_1}\right)^{(2t+1)s+j+1}\left(\frac{a_1}{b_0}\right)^{(2t+1)s+j-t}$$

$$+\sum_{l=0}^{m}\beta_{2(2tl+j+l-t)+1}u_0^{m-l+1}u_1^{m-l}$$
$$\times\left(\frac{a_1}{b_0}\right)^{(j+1)(m+1-l)+(2t+1)\frac{m(m+1)-l(l-1)}{2}}\left(\frac{a_0}{b_1}\right)^{(t+j+1)(m-l)+(2t+1)\frac{m(m-1)-l(l-1)}{2}}$$
$$+\sum_{l=0}^{m}\beta_{2(2tl+j+l+1)}u_0^{m-l}u_1^{m-l}\left(\frac{a_1}{b_0}\right)^{(j+1)(m-l)+(2t+1)\frac{m(m+1)-l(l+1)}{2}}$$
$$\times\left(\frac{a_0}{b_1}\right)^{(t+j+1)(m-l)+(2t+1)\frac{m(m-1)-l(l-1)}{2}}$$

$$y_{4tm+2m+2j+1}=y_{2j-4t-1}\prod_{s=0}^{m}\frac{a_0b_0}{d}\left(\frac{b_0}{a_1}\right)^{(2t+1)s+j}\left(\frac{b_1}{a_0}\right)^{(2t+1)s+j-t}$$
$$+\sum_{l=0}^{m}\alpha_{2(2tl+j+l-t)}v_0^{m-l}v_1^{m-l+1}$$
$$\times\left(\frac{b_1}{a_0}\right)^{(t+j+1)(m-l)+(2t+1)\frac{m(m-1)-l(l-1)}{2}}\left(\frac{b_0}{a_1}\right)^{j(m-l+1)+(2t+1)\frac{m(m+1)-l(l-1)}{2}}$$
$$+\sum_{l=0}^{m}\alpha_{2(2tl+j+l)+1}v_0^{m-l}v_1^{m-l}\left(\frac{b_1}{a_0}\right)^{(t+j+1)(m-l)+(2t+1)\frac{m(m-1)-l(l+1)}{2}}$$
$$\times\left(\frac{b_0}{a_1}\right)^{j(m-l)+(2t+1)\frac{m(m+1)-l(l+1)}{2}}$$

$$y_{4tm+2m+2j+2}=y_{2j-4t}\prod_{s=0}^{m}\frac{a_0b_0}{d}\left(\frac{b_0}{a_1}\right)^{(2t+1)s+j+1}\left(\frac{b_1}{a_0}\right)^{(2t+1)s+j-t}$$
$$+\sum_{l=0}^{m}\alpha_{2(2tl+j+l-t)+1}v_0^{m-l+1}v_1^{m-l}\left(\frac{b_1}{a_0}\right)^{(j+1)(m-l+1)+(2t+1)\frac{m(m+1)-l(l-1)}{2}}$$
$$\times\left(\frac{b_0}{a_1}\right)^{(t+j+1)(m-l)+(2t+1)\frac{m(m-1)-l(l-1)}{2}}+\sum_{l=0}^{m}\alpha_{2(2tl+j+l+1)}v_0^{m-l}v_1^{m-l}$$
$$\times\left(\frac{b_1}{a_0}\right)^{(j+1)(m-l)+(2t+1)\frac{m(m+1)-l(l+1)}{2}}\left(\frac{b_0}{a_1}\right)^{(t+j+1)(m-l)+(2t+1)\frac{m(m-1)-l(l-1)}{2}}$$

c) Suppose $k=(2t)$ $(t=1,2,\dots)$. Similarly, iterating the right-hand side of equations in (9), we have

$$x_{4tm+2j} \quad = \quad x_{2j-4t}\prod_{s=0}^{m}u_{2(2ts+j)}u_{2(2ts+j-t)}$$

$$+\sum_{l=0}^{2m+1}\beta_{2(tl+j-t)}\prod_{s=l+1}^{2m+1}u_{2(ts+j-t)}, \tag{19}$$

$$x_{4tm+2j+1} = x_{2j-4t+1}\prod_{s=0}^{m}u_{2(2ts+j)+1}u_{2(2ts+j-t)+1} + \sum_{l=0}^{2m+1}\beta_{2(tl+j-t)+1}\prod_{s=l+1}^{2m+1}u_{2(ts+j-t)+1}, \tag{20}$$

$$y_{4tm+2j} = y_{2j-4t}\prod_{s=0}^{m}v_{2(2ts+j)}v_{2(2ts+j-t)} + \sum_{l=0}^{2m+1}\alpha_{2(tl+j-t)}\prod_{s=l+1}^{2m+1}v_{2(ts+j-t)}, \tag{21}$$

$$y_{4tm+2j+1} = y_{2j-4t+1}\prod_{s=0}^{m}v_{2(2ts+j)+1}v_{2(2ts+j-t)+1} + \sum_{l=0}^{2m+1}\alpha_{2(tl+j-t)+1}\prod_{s=l+1}^{2m+1}v_{2(ts+j-t)+1}, \tag{22}$$

where $j \in \{t, t+1, ..., 3t-1\}$. Using (8) in (19)-(22), we get

$$x_{4tm+2j} = x_{2j-4t}\prod_{s=0}^{m}u_0^2\left(\frac{a_1}{b_0}\right)^{4ts+2j-t} + \sum_{l=0}^{2m+1}\beta_{2(tl+j-t)}u_0^{2m-l+1} \times\left(\frac{a_1}{b_0}\right)^{j(2m-l+1)+t\frac{2m(2m+1)-l(l-1)}{2}}$$

$$x_{4tm+2j+1} = x_{2j-4t+1}\prod_{s=0}^{m}u_1^2\left(\frac{a_0}{b_1}\right)^{4ts+2j-t} + \sum_{l=0}^{2m+1}\beta_{2(tl+j-t)+1}u_1^{2m-l+1} \times\left(\frac{a_0}{b_1}\right)^{j(2m-l+1)+t\frac{2m(2m+1)-l(l-1)}{2}}$$

$$y_{4tm+2j} = y_{2j-4t}\prod_{s=0}^{m}v_0^2\left(\frac{b_1}{a_0}\right)^{4ts+2j-t} + \sum_{l=0}^{2m+1}\alpha_{2(tl+j-t)}v_0^{2m-l+1} \times\left(\frac{b_1}{a_0}\right)^{j(2m-l+1)+t\frac{2m(2m+1)-l(l-1)}{2}}$$

$$y_{4tm+2j+1} = y_{2j-4t+1}\prod_{s=0}^{m}v_1^2\left(\frac{b_0}{a_1}\right)^{4ts+2j-t} + \sum_{l=0}^{2m+1}\alpha_{2(tl+j-t)+1}v_1^{2m-l+1}$$

$$\times \quad \left(\frac{b_0}{a_1}\right)^{j(2m-l+1)+t\frac{2m(2m+1)-l(l-1)}{2}},$$

where $u_0=\frac{x_0-\beta_0}{x_{-k}}$, $u_1=\frac{x_1-\beta_1}{x_{1-k}}$, $v_0=\frac{y_0-\alpha_0}{y_{-k}}$, $v_1=\frac{y_1-\alpha_1}{y_{1-k}}$, $d=u_0u_1$ and $v_0v_1=\frac{a_0b_0}{d}$.

The previous computations prove the next theorem.

Theorem 3. *Let$\{(x_n,y_n)\}_{n\geq -k}$ be a well-defined solution of system (3). Then, we get the next formulas:*

(a) If $k=1$, for all $n\in N_0$, then we get

$$\begin{aligned}x_{2n-1} &= x_{-1}\prod_{t=0}^{n-1}d\left(\frac{a_0}{b_1}\right)^t\left(\frac{a_1}{b_0}\right)^t+\sum_{r=0}^{n-1}\beta_0u_1\left(\frac{a_0}{b_1}\right)^r\prod_{t=r+1}^{n-1}d\left(\frac{a_0}{b_1}\right)^t\left(\frac{a_1}{b_0}\right)^t\\ &+ \sum_{r=0}^{n-1}\beta_1\prod_{t=r+1}^{n-1}d\left(\frac{a_0}{b_1}\right)^t\left(\frac{a_1}{b_0}\right)^t,\end{aligned}$$

$$\begin{aligned}x_{2n} &= x_0\prod_{t=0}^{n-1}d\left(\frac{a_0}{b_1}\right)^t\left(\frac{a_1}{b_0}\right)^{t+1}+\sum_{r=0}^{n-1}\beta_0\prod_{t=r+1}^{n-1}d\left(\frac{a_0}{b_1}\right)^t\left(\frac{a_1}{b_0}\right)^{t+1}\\ &+ \sum_{r=0}^{n-1}\beta_1u_0\left(\frac{a_1}{b_0}\right)^{r+1}\prod_{t=r+1}^{n-1}d\left(\frac{a_0}{b_1}\right)^t\left(\frac{a_1}{b_0}\right)^{t+1},\end{aligned}$$

$$\begin{aligned}y_{2n-1} &= y_{-1}\prod_{t=0}^{n-1}\frac{a_0b_0}{d}\left(\frac{b_0}{a_1}\right)^t\left(\frac{b_1}{a_0}\right)^t+\sum_{r=0}^{n-1}\alpha_0v_1\left(\frac{b_0}{a_1}\right)^r\prod_{t=r+1}^{n-1}\frac{a_0b_0}{d}\left(\frac{b_0}{a_1}\right)^t\left(\frac{b_1}{a_0}\right)^t\\ &+ \sum_{r=0}^{n-1}\alpha_1\prod_{t=r+1}^{n-1}\frac{a_0b_0}{d}\left(\frac{b_0}{a_1}\right)^t\left(\frac{b_1}{a_0}\right)^t,\end{aligned}$$

$$\begin{aligned}y_{2n} &= y_0\prod_{t=0}^{n-1}\frac{a_0b_0}{d}\left(\frac{b_0}{a_1}\right)^t\left(\frac{b_1}{a_0}\right)^{t+1}+\sum_{r=0}^{n-1}\alpha_0\prod_{t=r+1}^{n-1}\frac{a_0b_0}{d}\left(\frac{b_0}{a_1}\right)^t\left(\frac{b_1}{a_0}\right)^{t+1}\\ &+ \sum_{r=0}^{n-1}\alpha_1v_0\left(\frac{b_1}{a_0}\right)^{r+1}\prod_{t=r+1}^{n-1}\frac{a_0b_0}{d}\left(\frac{b_0}{a_1}\right)^t\left(\frac{b_1}{a_0}\right)^{t+1}.\end{aligned}$$

b) If $k=2t+1$ $(t=1,2,\dots)$, for all $m\in N_0$, we have

$$\begin{aligned}x_{4tm+2m+2j+1} &= x_{2j-4t-1}\prod_{s=0}^{m}d\left(\frac{a_0}{b_1}\right)^{(2t+1)s+j}\left(\frac{a_1}{b_0}\right)^{(2t+1)s+j-t}\\ &+\sum_{l=0}^{m}\beta_{2(2tl+j+l-t)}u_0^{m-l}u_1^{m-l+1}\left(\frac{a_1}{b_0}\right)^{(t+j+1)(m-l)+(2t+1)\frac{m(m-1)-l(l-1)}{2}}\end{aligned}$$

$$\times\left(\frac{a_0}{b_1}\right)^{j(m-l+1)+(2t+1)\frac{m(m+1)-l(l-1)}{2}}$$

$$+\sum_{l=0}^{m}\beta_{2(2tl+j+l)+1}u_0^{m-l}u_1^{m-l}\left(\frac{a_1}{b_0}\right)^{(t+j+1)(m-l)+(2t+1)\frac{m(m-1)-l(l+1)}{2}}$$

$$\times\left(\frac{a_0}{b_1}\right)^{j(m-l)+(2t+1)\frac{m(m+1)-l(l+1)}{2}}$$

$$x_{4tm+2m+2j+2}=x_{2j-4t}\prod_{s=0}^{m}d\left(\frac{a_0}{b_1}\right)^{(2t+1)s+j+1}\left(\frac{a_1}{b_0}\right)^{(2t+1)s+j-t}$$

$$+\sum_{l=0}^{m}\beta_{2(2tl+j+l-t)+1}u_0^{m-l+1}u_1^{m-l}$$

$$\times\left(\frac{a_1}{b_0}\right)^{(j+1)(m-l+1)+(2t+1)\frac{m(m+1)-l(l-1)}{2}}\left(\frac{a_0}{b_1}\right)^{(t+j+1)(m-l)+(2t+1)\frac{m(m-1)-l(l-1)}{2}}$$

$$+\sum_{l=0}^{m}\beta_{2(2tl+j+l+1)}u_0^{m-l}u_1^{m-l}\left(\frac{a_1}{b_0}\right)^{(j+1)(m-l)+(2t+1)\frac{m(m+1)-l(l+1)}{2}}$$

$$\times\left(\frac{a_0}{b_1}\right)^{(t+j+1)(m-l)+(2t+1)\frac{m(m-1)-l(l-1)}{2}}$$

$$\begin{aligned}y_{4tm+2m+2j+1} &= y_{2j-4t-1}\prod_{s=0}^{m}\frac{a_0b_0}{d}\left(\frac{b_0}{a_1}\right)^{(2t+1)s+j}\left(\frac{b_1}{a_0}\right)^{(2t+1)s+j-t}\\ &\quad+\sum_{l=0}^{m}\alpha_{2(2tl+j+l-t)}v_0^{m-l}v_1^{m-l+1}\\ &\times\left(\frac{b_1}{a_0}\right)^{(t+j+1)(m-l)+(2t+1)\frac{m(m-1)-l(l-1)}{2}}\left(\frac{b_0}{a_1}\right)^{j(m-l+1)+(2t+1)\frac{m(m+1)-l(l-1)}{2}}\\ &+\sum_{l=0}^{m}\alpha_{2(2tl+j+l)+1}v_0^{m-l}v_1^{m-l}\left(\frac{b_1}{a_0}\right)^{(t+j+1)(m-l)+(2t+1)\frac{m(m-1)-l(l+1)}{2}}\\ &\times\left(\frac{b_0}{a_1}\right)^{j(m-l)+(2t+1)\frac{m(m+1)-l(l+1)}{2}}\end{aligned}$$

$$y_{4tm+2m+2j+2}=y_{2j-4t}\prod_{s=0}^{m}\frac{a_0b_0}{d}\left(\frac{b_0}{a_1}\right)^{(2t+1)s+j+1}\left(\frac{b_1}{a_0}\right)^{(2t+1)s+j-t}$$

$$+\sum_{l=0}^{m}\alpha_{2(2tl+j+l-t)+1}v_0^{m-l+1}v_1^{m-l}\left(\frac{b_1}{a_0}\right)^{(j+1)(m-l+1)+(2t+1)\frac{m(m+1)-l(l-1)}{2}}$$
$$\times\left(\frac{b_0}{a_1}\right)^{(t+j+1)(m-l)+(2t+1)\frac{m(m-1)-l(l-1)}{2}}+\sum_{l=0}^{m}\alpha_{2(2tl+j+l+1)}v_0^{m-l}v_1^{m-l}$$
$$\times\left(\frac{b_1}{a_0}\right)^{(j+1)(m-l)+(2t+1)\frac{m(m+1)-l(l+1)}{2}}\left(\frac{b_0}{a_1}\right)^{(t+j+1)(m-l)+(2t+1)\frac{m(m-1)-l(l-1)}{2}}$$

c) If $k=2t\,(t=1,2,\dots)$, for all $m\in N_0$, we have

$$\begin{aligned}
x_{4tm+2j} &= x_{2j-4t}\prod_{s=0}^{m}u_0^2\left(\frac{a_1}{b_0}\right)^{4ts+2j-t}+\sum_{l=0}^{2m+1}\beta_{2(tl+j-t)}u_0^{2m-l+1}\\
&\times\left(\frac{a_1}{b_0}\right)^{j(2m-l+1)+t\frac{2m(2m+1)-l(l-1)}{2}}\\
x_{4tm+2j+1} &= x_{2j-4t+1}\prod_{s=0}^{m}u_1^2\left(\frac{a_0}{b_1}\right)^{4ts+2j-t}+\sum_{l=0}^{2m+1}\beta_{2(tl+j-t)+1}u_1^{2m-l+1}\\
&\times\left(\frac{a_0}{b_1}\right)^{j(2m-l+1)+t\frac{2m(2m+1)-l(l-1)}{2}}\\
y_{4tm+2j} &= y_{2j-4t}\prod_{s=0}^{m}v_0^2\left(\frac{b_1}{a_0}\right)^{4ts+2j-t}+\sum_{l=0}^{2m+1}\alpha_{2(tl+j-t)}v_0^{2m-l+1}\\
&\times\left(\frac{b_1}{a_0}\right)^{j(2m-l+1)+t\frac{2m(2m+1)-l(l-1)}{2}}\\
y_{4tm+2j+1} &= y_{2j-4t+1}\prod_{s=0}^{m}v_1^2\left(\frac{b_0}{a_1}\right)^{4ts+2j-t}+\sum_{l=0}^{2m+1}\alpha_{2(tl+j-t)+1}v_1^{2m-l+1}\\
&\times\left(\frac{b_0}{a_1}\right)^{j(2m-l+1)+t\frac{2m(2m+1)-l(l-1)}{2}},
\end{aligned}$$

where $u_0=\frac{x_0-\beta_0}{x_{-k}}$, $u_1=\frac{x_1-\beta_1}{x_{1-k}}$, $v_0=\frac{y_0-\alpha_0}{y_{-k}}$, $v_1=\frac{y_1-\alpha_1}{y_{1-k}}$, $d=u_0u_1$ and $v_0v_1=\frac{a_0b_0}{d}$.

3. The case $a_0=b_1$ and $a_1=b_0$

The aim of in this section is to investigate the asymptotic behavior and periodicity of well-defined solutions of system (3) in the case $a_0=b_1$ and $a_1=b_0$. The following corollary gives us the solutions of system (3) when $a_0=b_1$ and $a_1=b_0$.

Corollary 4. *Suppose that $\{(x_n,y_n)\}_{n\geq -k}$ is a well-defined solution of system (3) when $a_0=b_1$ and $a_1=b_0$. Then, the solutions of system (3) are given by:*

a) If $k = 1$, for all $m \in N_0$, then we get

$$x_{2m-1} = \begin{cases} x_{-1} + (\beta_0 u_1 + \beta_1)\, m & \text{if } d = 1, \\ x_{-1} d^m + (\beta_0 u_1 + \beta_1)\, \frac{d^m - 1}{d-1} & \text{if } d \neq 1, \end{cases} \tag{23}$$

$$x_{2m} = \begin{cases} x_0 + (\beta_1 u_0 + \beta_0)\, m & \text{if } d = 1, \\ x_0 d^m + (\beta_1 u_0 + \beta_0)\, \frac{d^m - 1}{d-1} & \text{if } d \neq 1, \end{cases} \tag{24}$$

$$y_{2m-1} = \begin{cases} y_{-1} + (\alpha_0 v_1 + \alpha_1)\, m & \text{if } a_0 b_0 = d, \\ y_{-1} \left(\frac{a_0 b_0}{d}\right)^m + (\alpha_0 v_1 + \alpha_1)\, \frac{\left(\frac{a_0 b_0}{d}\right)^m - 1}{\frac{a_0 b_0}{d} - 1} & \text{if } a_0 b_0 \neq d, \end{cases} \tag{25}$$

$$y_{2m} = \begin{cases} y_0 + (\alpha_1 v_0 + \alpha_0)\, m & \text{if } a_0 b_0 = d, \\ y_0 \left(\frac{a_0 b_0}{d}\right)^m + (\alpha_1 v_0 + \alpha_0)\, \frac{\left(\frac{a_0 b_0}{d}\right)^m - 1}{\frac{a_0 b_0}{d} - 1} & \text{if } a_0 b_0 \neq d, \end{cases} \tag{26}$$

b) If $k = 2t + 1\, (t = 1, 2, \dots)$, for all $m \in N_0$ and $j \in \{t, t+1, ..., 3t\}$, we have

$$x_{4tm+2m+2j+1} = \begin{cases} x_{2j-4t-1} + (\beta_0 u_1 + \beta_1)\, (m+1) & \text{if } d = 1, \\ x_{2j-4t-1} d^{m+1} + (\beta_0 u_1 + \beta_1) \left(\frac{d^{m+1} - 1}{d-1}\right) & \text{if } d \neq 1, \end{cases} \tag{27}$$

$$x_{4tm+2m+2j+2} = \begin{cases} x_{2j-4t} + (\beta_1 u_0 + \beta_0)\, (m+1) & \text{if } d = 1, \\ x_{2j-4t} d^{m+1} + (\beta_1 u_0 + \beta_0) \left(\frac{d^{m+1} - 1}{d-1}\right) & \text{if } d \neq 1, \end{cases} \tag{28}$$

$$y_{4tm+2m+2j+1} = \begin{cases} y_{2j-4t-1} + (\alpha_0 v_1 + \alpha_1)\, (m+1) & \text{if } d = a_0 b_0, \\ y_{2j-4t-1} \left(\frac{a_o b_o}{d}\right)^{m+1} + (\alpha_0 v_1 + \alpha_1) \left(\frac{\left(\frac{a_0 b_0}{d}\right)^{m+1} - 1}{\frac{a_0 b_0}{d} - 1}\right) & \text{if } d \neq a_0 b_0, , \end{cases} \tag{29}$$

$$y_{4tm+2m+2j+2} = \begin{cases} y_{2j-4t} + (\alpha_1 v_0 + \alpha_0)\, (m+1) & \text{if } d = a_0 b_0 \\ y_{2j-4t} \left(\frac{a_0 b_0}{d}\right)^{m+1} + (\alpha_1 v_0 + \alpha_0) \left(\frac{\left(\frac{a_0 b_0}{d}\right)^{m+1} - 1}{\frac{a_0 b_0}{d} - 1}\right) & \text{if } d \neq a_0 b_0, \end{cases} \tag{30}$$

c) If $k = 2t\, (t = 1, 2, \dots)$, for all $m \in N_0$ and $j \in \{t, t+1, ..., 3t-1\}$, we have

$$x_{4tm+2j+1} = \begin{cases} x_{2j-4t+1} + \beta_1\, (2m+2) & \text{if } u_1 = 1, \\ x_{2j-4t+1} u_1^{2m+2} + \beta_1 \left(\frac{(u_1)^{2m+2} - 1}{u_1 - 1}\right) & \text{if } u_1 \neq 1, \end{cases} \tag{31}$$

$$x_{4tm+2j} = \begin{cases} x_{2j-4t} + \beta_0\, (2m+2) & \text{if } u_0 = 1, \\ x_{2j-4t}\, (u_0)^{2m+2} + \beta_0 \left(\frac{(u_0)^{2m+2} - 1}{u_0 - 1}\right) & \text{if } u_0 \neq 1, \end{cases} \tag{32}$$

$$y_{4tm+2j+1} = \begin{cases} y_{2j-4t+1} + \alpha_1\, (2m+2) & \text{if } v_1 = 1, \\ y_{2j-4t+1}\, (v_1)^{2m+2} + \alpha_1 \left(\frac{(v_1)^{2m+2} - 1}{v_1 - 1}\right) & \text{if } v_1 \neq 1, \end{cases} \tag{33}$$

$$y_{4tm+2j}=\begin{cases} y_{2j-4t}+\alpha_0\left(2m+2\right) & \text{if } v_0=1,\\ y_{2j-4t}\left(v_0\right)^{2m+2}+\alpha_0\left(\frac{(v_0)^{2m+2}-1}{v_0-1}\right) & \text{if } v_0\neq 1. \end{cases} \tag{34}$$

where $u_0=\frac{x_0-\beta_0}{x_{-k}}$, $u_1=\frac{x_1-\beta_1}{x_{1-k}}$, $v_0=\frac{y_0-\alpha_0}{y_{-k}}$, $v_1=\frac{y_1-\alpha_1}{y_{1-k}}$, $d=u_0u_1$ *and* $v_0v_1=\frac{a_0b_0}{d}$.

The next theorem gives the limiting properties of solutions of system (3) in the case $a_0=b_1$ and $a_1=b_0$.

Theorem 5. *Suppose that* $\{(x_n,y_n)\}_{n\geq -k}$ *is a well-defined solution of system (3) with* $a_0=b_1$, $a_1=b_0$. *Then, the next statements are true.*

(1) *Let* $k=1$.

(a) *If* $(d-1)\,x_{-1}+(\beta_0u_1+\beta_1)\neq 0$, *then we get*

$$\lim_{n\to\infty}|x_{2n-1}|=\begin{cases}\left|\frac{\beta_0u_1+\beta_1}{d-1}\right| & \text{If } |d|<1\ ,\\ \infty & \text{if } |d|>1\ .\end{cases}$$

Otherwise, if $(d-1)\,x_{-1}+(\beta_0u_1+\beta_1)=0$ *and* $d\neq 1$, *then* $x_{2n-1}=x_{-1}$ *for all* $n\in N_0$.

(b) *If* $\beta_0u_1+\beta_1\neq 0$ *and* $d=1$, *then* $|x_{2n-1}|\to\infty$, *as* $n\to\infty$. *Otherwise, if* $\beta_0u_1+\beta_1=0$ *and* $d=1$, *then* $x_{2n-1}=x_{-1}$, *for all* $n\in N_0$.

(c) *If* $(d-1)\,x_0+(\beta_1u_0+\beta_0)\neq 0$, *then we have*

$$\lim_{n\to\infty}|x_{2n}|=\begin{cases}\left|\frac{\beta_1u_0+\beta_0}{d-1}\right| & \text{if } |d|<1,\\ \infty & \text{if } |d|>1.\end{cases}$$

Otherwise, if $(d-1)\,x_0+(\beta_1u_0+\beta_0)=0$ *and* $d\neq 1$, *then* $x_{2n}=x_0$ *for all* $n\in N_0$.

(d) *If* $\beta_1u_0+\beta_0\neq 0$ *and* $d=1$, *then* $|x_{2n}|\to\infty$, *as* $n\to\infty$. *Otherwise, if* $\beta_1u_0+\beta_0=0$ *and* $d=1$, *then* $x_{2n}=x_0$, *for all* $n\in N_0$.

(e) *If* $\left(\frac{a_0b_0}{d}-1\right)y_{-1}+(\alpha_0v_1+\alpha_1)\neq 0$, *then we get*

$$\lim_{n\to\infty}|y_{2n-1}|=\begin{cases}\left|\frac{(\alpha_0v_1+\alpha_1)d}{d-a_0b_0}\right| & \text{if } |d|>a_0b_0,\\ \infty & \text{if } |d|<a_0b_0.\end{cases}$$

Otherwise, if $\left(\frac{a_0b_0}{d}-1\right)y_{-1}+(\alpha_0v_1+\alpha_1)=0$ *and* $d\neq a_0b_0$, *then* $y_{2n-1}=y_{-1}$ *for all* $n\in N_0$.

(f) *If* $\alpha_0v_1+\alpha_1\neq 0$ *and* $d=a_0b_0$, *then* $|y_{2n-1}|\to\infty$, *as* $n\to\infty$. *Otherwise, if* $\alpha_0v_1+\alpha_1=0$ *and* $d=a_0b_0$, *then* $y_{2n-1}=y_{-1}$, *for all* $n\in N_0$.

(g) *If* $\left(\frac{a_0b_0}{d}-1\right)y_0+(\alpha_1v_0+\alpha_0)\neq 0$, *then we get*

$$\lim_{n\to\infty}|y_{2n}|=\begin{cases}\left|\frac{(\alpha_1v_0+\alpha_0)d}{d-a_0b_0}\right| & \text{if } |d|>a_0b_0\\ \infty & \text{if } |d|<a_0b_0\end{cases}$$

Otherwise, if $\left(\frac{a_0b_0}{d}-1\right)y_0+(\alpha_1v_0+\alpha_0)=0$ *and* $d\neq a_0b_0$, *then* $y_{2n}=y_0$ *for all* $n\in N_0$.

(h) *If* $\alpha_1v_0+\alpha_0\neq 0$ *and* $d=a_0b_0$, *then* $|y_{2n}|\to\infty$, *as* $n\to\infty$. *Otherwise, if* $\alpha_1v_0+\alpha_0=0$ *and* $d=a_0b_0$, *then* $y_{2n}=y_0$, *for all* $n\in N_0$.

(2) *Let* $k=2t+1\,(t=1,2,\dots)$ *and* $j\in\{t,t+1,...,3t\}$.

(a) *If* $(d-1)\,x_{2j-4t-1}+(\beta_0u_1+\beta_1)\neq 0$, *then we have*

$$\lim_{m\to\infty}|x_{4tm+2m+2j+1}|=\begin{cases}\left|\frac{\beta_0u_1+\beta_1}{d-1}\right| & \text{if } |d|<1,\\ \infty & \text{if } |d|>1.\end{cases}$$

Otherwise, if $(d-1)\,x_{2j-4t-1}+(\beta_0u_1+\beta_1)=0$ *and* $d\neq 1$, *then* $x_{4tm+2m+2j+1}=x_{2j-4t-1}$ *for all* $m\in N_0$.

(b) *If* $\beta_0u_1+\beta_1\neq 0$ *and* $d=1$, *then* $|x_{4tm+2m+2j+1}|\to\infty$, *as* $m\to\infty$. *Otherwise, if* $\beta_0u_1+\beta_1=0$ *and* $d=1$, *then* $x_{4tm+2m+2j+1}=x_{2j-4t-1}$, *for all* $m\in N_0$.

(c) *If* $(d-1)\,x_{2j-4t}+(\beta_1u_0+\beta_0)\neq 0$, *then we get*

$$\lim_{m\to\infty}|x_{4tm+2m+2j+2}|=\begin{cases}\left|\frac{\beta_1u_0+\beta_0}{d-1}\right| & \text{if } |d|<1,\\ \infty & \text{if } |d|>1\ .\end{cases}$$

Otherwise, if $(d-1)\,x_{2j-4t}+(\beta_1u_0+\beta_0)=0$ *and* $d\neq 1$, *then* $x_{4tm+2m+2j+2}=x_{2j-4t}$ *for all* $m\in N_0$.

(d) *If* $\beta_1u_0+\beta_0\neq 0$ *and* $d=1$, *then* $|x_{4tm+2m+2j+2}|\to\infty$, *as* $m\to\infty$. *Otherwise, if* $\beta_1u_0+\beta_0=0$ *and* $d=1$, *then* $x_{4tm+2m+2j+2}=x_{2j-4t}$, *for all* $m\in N_0$.

(e) *If* $\left(\frac{a_0b_0}{d}-1\right)y_{2j-4t-1}+(\alpha_0v_1+\alpha_1)\neq 0$, *then we have*

$$\lim_{m\to\infty}|y_{4tm+2m+2j+1}|=\begin{cases}\left|\frac{(\alpha_0v_1+\alpha_1)d}{d-a_0b_0}\right| & \text{if } |d|>a_0b_0,\\ \infty & \text{if } |d|<a_0b_0\ .\end{cases}$$

Otherwise, if $\left(\frac{a_0b_0}{d}-1\right)y_{2j-4t-1}+(\alpha_0v_1+\alpha_1)=0$ *and* $d\neq a_0b_0$, *then* $y_{4tm+2m+2j+1}=y_{2j-4t-1}$ *for all* $m\in N_0$.

(f) *If* $\alpha_0v_1+\alpha_1\neq 0$ *and* $d=a_0b_0$, *then* $|y_{4tm+2m+2j+1}|\to\infty$, *as* $m\to\infty$. *Otherwise, if* $\alpha_0v_1+\alpha_1=0$ *and* $d=a_0b_0$, *then* $y_{4tm+2m+2j+1}=y_{2j-4t-1}$, *for all* $m\in N_0$.

(g) *If* $\left(\frac{a_0b_0}{d}-1\right)y_{2j-4t}+(\alpha_1v_0+\alpha_0)\neq 0$, *then we get*

$$\lim_{m\to\infty}|y_{4tm+2m+2j+2}|=\begin{cases}\left|\frac{(\alpha_1v_0+\alpha_0)d}{d-a_0b_0}\right| & \text{if } |d|>a_0b_0,\\ \infty & \text{if } |d|<a_0b_0.\end{cases}$$

Otherwise, if $\left(\frac{a_0b_0}{d}-1\right)y_{2j-4t}+(\alpha_1v_0+\alpha_0)=0$ *and* $d\neq a_0b_0$, *then* $y_{4tm+2m+2j+2}=y_{2j-4t}$ *for all* $m\in N_0$.

(h) *If* $\alpha_1 v_0+\alpha_0 \neq 0$ *and* $d=a_0b_0$, *then* $|y_{4tm+2m+2j+2}| \to \infty$, *as* $m \to \infty$. *Otherwise, if* $\alpha_1 v_0+\alpha_0 = 0$ *and* $d=a_0b_0$, *then* $y_{4tm+2m+2j+2} = y_{2j-4t}$, *for all* $m \in N_0$.

(3) *Let* $k=2t\,(t=1,2,\dots)$ *and* $j \in \{t, t+1, ..., 3t-1\}$.

(a) *If* $(u_1-1)\,x_{2j-4t+1}+\beta_1 \neq 0$, *then we have*

$$\lim_{m\to\infty} |x_{4tm+2j+1}| = \begin{cases} \left|\frac{\beta_1}{u_1-1}\right| & \text{if } |u_1|<1, \\ \infty & \text{if } |u_1|>1. \end{cases}$$

Otherwise, if $(u_1-1)\,x_{2j-4t+1}+\beta_1 = 0$ *and* $u_1 \neq 1$, *then* $x_{4tm+2j+1} = x_{2j-4t+1}$ *for all* $m \in N_0$.

(b) *If* $\beta_1 \neq 0$ *and* $u_1 = 1$, *then* $|x_{4tm+2j+1}| \to \infty$, *as* $m \to \infty$. *Otherwise, if* $\beta_1 = 0$ *and* $u_1 = 1$, *then* $x_{4tm+2j+1} = x_{2j-4t+1}$, *for all* $m \in N_0$.

(c) *If* $(u_0-1)\,x_{2j-4t}+\beta_0 \neq 0$, *then we get*

$$\lim_{m\to\infty} |x_{4tm+2j}| = \begin{cases} \left|\frac{\beta_0}{u_0-1}\right| & \text{if } |u_0|<1, \\ \infty & \text{if } |u_0|>1. \end{cases}$$

Otherwise, if $(u_0-1)\,x_{2j-4t}+\beta_0 = 0$ *and* $u_0 \neq 1$, *then* $x_{4tm+2j} = x_{2j-4t}$ *for all* $m \in N_0$.

(d) *If* $\beta_0 \neq 0$ *and* $u_0 = 1$, *then* $|x_{4tm+2j}| \to \infty$, *as* $m \to \infty$. *Otherwise, if* $\beta_0 = 0$ *and* $u_0 = 1$, *then* $x_{4tm+2j} = x_{2j-4t}$, *for all* $m \in N_0$.

(e) *If* $(v_1-1)\,y_{2j-4t+1}+\alpha_1 \neq 0$, *then we have*

$$\lim_{m\to\infty} |y_{4tm+2j+1}| = \begin{cases} \left|\frac{\alpha_1}{v_1-1}\right| & \text{if } |v_1|<1, \\ \infty & \text{if } |v_1|>1. \end{cases}$$

Otherwise, if $(v_1-1)\,y_{2j-4t+1}+\alpha_1 = 0$ *and* $v_1 \neq 1$, *then* $y_{4tm+2j+1} = y_{2j-4t+1}$ *for all* $m \in N_0$.

(f) *If* $\alpha_1 \neq 0$ *and* $v_1 = 1$, *then* $|y_{4tm+2j+1}| \to \infty$, *as* $m \to \infty$. *Otherwise, if* $\alpha_1 = 0$ *and* $v_1 = 1$, *then* $y_{4tm+2j+1} = y_{2j-4t+1}$, *for all* $m \in N_0$.

(g) *If* $(v_0-1)\,y_{2j-4t}+\alpha_0 \neq 0$, *then we get*

$$\lim_{m\to\infty} |y_{4tm+2j}| = \begin{cases} \left|\frac{\alpha_0}{v_0-1}\right| & \text{if } |v_0|<1, \\ \infty & \text{if } |v_0|>1. \end{cases}$$

Otherwise, if $(v_0-1)\,y_{2j-4t}+\alpha_0 = 0$ *and* $v_0 \neq 1$, *then* $y_{4tm+2j} = y_{2j-4t}$ *for all* $m \in N_0$.

(h) *If* $\alpha_0 \neq 0$ *and* $v_0 = 1$, *then* $|y_{4tm+2j}| \to \infty$, *as* $m \to \infty$. *Otherwise, if* $\alpha_0 = 0$ *and* $v_0 = 1$, *then* $y_{4tm+2j} = y_{2j-4t}$, *for all* $m \in N_0$.

Proof. We will only prove the items (a) and (b) for $k=2t+1(t=1,2,...)$ and $j \in \{t, t+1, ..., 3t\}$ since the other cases can be proved similarly.

(a) Assume that $(d-1)\,x_{2j-4t-1}+(\beta_0 u_1+\beta_1) \neq 0$. It is easy to see from Corollary 4 that $x_{4tm+2m+2j+1} \neq 0$. Clearly, if $|d|<1$, then $|d|^{m+1} \to 0$

as $m \to \infty$. On the other hand, if $|d| > 1$, then $|d|^{m+1} \to \infty$ as $m \to \infty$. From Eq. (27), we get

$$\begin{aligned}
\lim_{m\to\infty} |x_{4tm+2m+2j+1}| &= \lim_{m\to\infty} \left| \frac{(d-1)\, x_{2j-4t-1} + (\beta_0 u_1 + \beta_1)}{d-1} d^{m+1} \right. \\
&\quad \left. + \left(\frac{(\beta_0 u_1 + \beta_1)}{1-d} \right) \right| \\
&= \left| \frac{(d-1)\, x_{2j-4t-1} + (\beta_0 u_1 + \beta_1)}{d-1} \lim_{m\to\infty} d^{m+1} \right. \\
&\quad \left. + \left(\frac{(\beta_0 u_1 + \beta_1)}{1-d} \right) \right| \\
&= \begin{cases} \left| \frac{(\beta_0 u_1 + \beta_1)}{d-1} \right| & \text{if } |d| < 1 \\ \infty & \text{if } |d| > 1 \end{cases}.
\end{aligned}$$

Now on the other hand $(d-1)\, x_{2j-4t-1} + (\beta_0 u_1 + \beta_1) = 0$ and $d \neq 1$. Then we have

$$\begin{aligned}
x_{4tm+2m+2j+1} &= x_{2j-4t-1} d^{m+1} + (\beta_0 u_1 + \beta_1) \left(\frac{d^{m+1} - 1}{d-1} \right) \\
&= x_{2j-4t-1} d^{m+1} + \left(\frac{d^{m+1} - 1}{d-1} \right) (-(d-1)\, x_{2j-4t-1}) \\
&= x_{2j-4t-1} d^{m+1} - \left(d^{m+1} - 1 \right) x_{2j-4t-1} \\
&= x_{2j-4t-1},
\end{aligned}$$

which completes the proof of (a).

(b) Let $d = 1$. If $\beta_0 u_1 + \beta_1 \neq 0$, then from Eq. (27) we get

$$x_{4tm+2m+2j+1} = x_{2j-4t-1} + (\beta_0 u_1 + \beta_1)(m+1) \neq 0.$$

Letting $m \to \infty$ in above equations implies that $|x_{4tm+2m+2j+1}| \to \infty$. On the other hand, If $\beta_0 u_1 + \beta_1 = 0$, then obviously, for all $m \in N_0$,

$$x_{4tm+2m+2j+1} = x_{2j-4t-1} + (\beta_0 u_1 + \beta_1)(m+1) = x_{2j-4t-1} + 0 = x_{2j-4t-1},$$

which completes the proof of (b).

□

Corollary 6. *Suppose that* $\{(x_n, y_n)\}_{n \geq -k}$ *is a well-defined solution of system (3) with* $a_0 = b_1$, $a_1 = b_0$, $u_0 u_1 = d$, $v_0 v_1 = \frac{a_0 b_0}{d}$. *Then, the next statements are true.*

(1) *Let* $k = 1$.

(a) *If* $d=-1$, *then for all* $m\in N_0$, *we get*

$$\begin{cases} x_{4m-1} &= x_{-1},\\ x_{4m} &= x_0,\\ x_{4m+1} &= -x_{-1}+(\beta_0u_1+\beta_1),\\ x_{4m+2} &= -x_0+(\beta_1u_0+\beta_0). \end{cases}$$

(b) *If* $d=-a_0b_0$, *then for all* $m\in N_0$, *we have*

$$\begin{cases} y_{4m-1} &= y_{-1},\\ y_{4m} &= y_0,\\ y_{4m+1} &= -y_{-1}+(\alpha_0v_1+\alpha_1),\\ y_{4m+2} &= -y_0+(\alpha_1v_0+\alpha_0). \end{cases}$$

(c) *If* $d=1, u_1=u_0=1, \beta_0+\beta_1=0$, *then for all* $m\in N_0$, *we get*

$$\begin{cases} x_{2m-1} &= x_{-1},\\ x_{2m} &= x_0. \end{cases}$$

(d) *If* $d=a_0b_0, v_1=v_0=1, \alpha_0+\alpha_1=0$, *then for all* $m\in N_0$, *we have*

$$\begin{cases} y_{2m-1} &= y_{-1},\\ y_{2m} &= y_0. \end{cases}$$

(2) *Let* $k=2t+1, (t=1,2,\dots)$ *and* $j\in\{t,t+1,\dots,3t\}$.

(a) *If* $d=-1$, *then for all* $m\in N_0$, *we have*

$$\begin{cases} x_{8tm+4m+2j+1} &= -x_{2j-4t-1}+(\beta_0u_1+\beta_1),\\ x_{8tm+4m+4t+2j+3} &= x_{2j-4t-1},\\ x_{8tm+4m+2j+2} &= -x_{2j-4t}+(\beta_1u_0+\beta_0),\\ x_{8tm+4m+4t+2j+4} &= x_{2j-4t}. \end{cases}$$

(b) *If* $d=-a_0b_0$, *then for all* $m\in N_0$, *we get*

$$\begin{cases} y_{8tm+4m+2j+1} &= -y_{2j-4t-1}+(\alpha_0v_1+\alpha_1),\\ y_{8tm+4m+4t+2j+3} &= y_{2j-4t-1},\\ y_{8tm+4m+2j+2} &= -y_{2j-4t}+(\alpha_1v_0+\alpha_0),\\ y_{8tm+4m+4t+2j+4} &= y_{2j-4t}. \end{cases}$$

(c) *If* $d=1$, $u_1=u_0=1$, $\beta_0+\beta_1=0$ *then for all* $m\in N_0$, *we have*

$$\begin{cases} x_{4tm+2m+2j+1} &= x_{2j-4t-1},\\ x_{4tm+2m+2j+2} &= x_{2j-4t}. \end{cases}$$

(d) *If* $d = a_0 b_0$, $v_1 = v_0 = 1$, $\alpha_0 + \alpha_1 = 0$ *then for all* $m \in N_0$, *we get*

$$\begin{cases} y_{4tm+2m+2j+1} & = y_{2j-4t-1}, \\ y_{4tm+2m+2j+2} & = y_{2j-4t}. \end{cases}$$

(3) *Let* $k = 2t, (t = 1, 2, \dots)$ *and* $j \in \{t, t+1, \dots, 3t-1\}$. *If* $u_1 = u_0 = v_1 = v_0 = -1$ *or* $u_1 = u_0 = v_1 = v_0 = 1$ *and* $\beta_1 = \beta_0 = \alpha_1 = \alpha_0 = 0$, *then for all* $m \in N_0$, *we have*

$$\begin{cases} x_{4tm+2j+1} & = x_{2j-4t+1}, \\ x_{4tm+2j} & = x_{2j-4t}, \\ y_{4tm+2j+1} & = y_{2j-4t+1}, \\ y_{4tm+2j} & = y_{2j-4t}. \end{cases}$$

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