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On the generalized Fibonacci and Lucas 2^k-ions

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Abstract: This study introduces the modified generalized Fibonacci and Lucas 2^k-ions which are the generalizations of several quaternions, octonions and higher order dimensional algebras. We give the generating functions, the Binet formulas and well-known identities such as Catalan's identity and Cassini's identity for the modified generalized Fibonacci and Lucas 2^k-ions.

Keywords: Modified generalized Fibonacci sequence, Modified generalized Lucas sequence, Recurrence relations, 2^k-ions.
2010 Mathematics Subject Classification: 11B39, 05A15.

1 Introduction

The Fibonacci and Lucas numbers arise in several areas such as mathematics, physics, computer science and related fields. The Fibonacci and Lucas numbers are defined by the recurrence relations, for $n \geq 0$,

$$F(0) = 0, \quad F(1) = 1, \quad F_{n+2} = F_{n+1} + F_n, \tag{1}$$

and

$$L(0) = 2, \quad L(1) = 1, \quad L_{n+2} = L_{n+1} + L_n, \tag{2}$$

respectively. For more information about the Fibonacci and Lucas numbers, we refer the readers to book [12]. Until this time, there have been a lot of applications and generalizations of the Fibonacci and Lucas numbers [1, 5, 6, 19–21]. For example, Falcon and Plaza found the general $k-$Fibonacci sequence $\{F_{k,n}\}_{n=0}^{\infty}$ by studying the recursive application of two geometrical transformations used in the well-known $4-$triangle longest-edge (4TLE) partition [6].

Furthermore, Yayenie [19] defined the modified generalized Fibonacci sequence as

$$Q_0 = 0, \quad Q_1 = 1, \quad Q_n = \begin{cases} aQ_{n-1} + cQ_{n-2} & \text{if } n \text{ is even} \\ bQ_{n-1} + dQ_{n-2} & \text{if } n \text{ is odd} \end{cases}, \quad n \geq 2, \tag{3}$$

where a, b, c and d are real numbers. Also he gave generating function, the generalized Binet formula and some basic identities for Q_n. By analogy to the studies [5] and [19], Bilgici [1] defined the bi-periodic Lucas numbers and modified generalized Lucas numbers and gave generating functions, the Binet formulas and some special identities for these sequences. He defined the modified generalized Lucas sequence as

$$U_0 = \frac{d+1}{d}, \quad U_1 = a, \quad U_n = \begin{cases} bU_{n-1} + dU_{n-2} & \text{if } n \text{ is even} \\ aU_{n-1} + cU_{n-2} & \text{if } n \text{ is odd} \end{cases}, \quad n \geq 2, \tag{4}$$

where a, b, c and d are real numbers. The generating functions of Q_n and U_n are given by

$$H(x) = \sum_{n=0}^{\infty} Q_n x^n = \frac{x\left(1 + ax - cx^2\right)}{1 - (ab + c + d)\, x^2 + cdx^4} \tag{5}$$

and

$$U(x) = \sum_{n=0}^{\infty} U_n x^n = \frac{1}{d}\left(\frac{d + 1 + adx - (ab + cd + c)\, x^2 + adx^3}{1 - (ab + c + d)\, x^2 + cdx^4}\right), \tag{6}$$

respectively. In addition, the Binet formulas of the sequences Q_n and U_n are also given by the following formulas:

$$Q_n = \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^{\lfloor \frac{n}{2} \rfloor} (\alpha + d - c)^{n - \lfloor \frac{n}{2} \rfloor} - \beta^{\lfloor \frac{n}{2} \rfloor} (\beta + d - c)^{n - \lfloor \frac{n}{2} \rfloor}}{\alpha - \beta}\right) \tag{7}$$

and

$$U_n = \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}} \left(\frac{(\alpha + d + 1)\, \alpha^{\lfloor \frac{n-1}{2} \rfloor} (\alpha + d - c)^{\lfloor \frac{n}{2} \rfloor} - (\beta + d + 1)\, \beta^{\lfloor \frac{n-1}{2} \rfloor} (\beta + d - c)^{\lfloor \frac{n}{2} \rfloor}}{\alpha - \beta}\right), \tag{8}$$

where $\alpha = \dfrac{ab + c - d + \sqrt{(ab + c - d)^2 + 4abd}}{2}$ and $\beta = \dfrac{ab + c - d - \sqrt{(ab + c - d)^2 + 4abd}}{2}$ are the roots of the polynomial $x^2 - (ab + c - d)\, x - abd = 0$ and $\xi(n) = n - 2\lfloor \frac{n}{2} \rfloor$ is the parity function which we use throughout the paper. Note that, we also assume that $\Delta = (ab + c - d)^2 + 4abd > 0$.

The Cayley–Dickson algebras are a sequence $\{A_0, A_1, A_2, \ldots\}$ of non-associative $\mathbb{R}-$algebras with involution. Every algebra A_k is built up from the previous one A_{k-1} with a procedure which destroy some algebra properties. The first five Cayley–Dickson algebras are familiar: $A_0 = \mathbb{R}$, $A_1 = \mathbb{C}$, $A_2 = \mathbb{H}$ (Quaternions), $A_3 = \mathbb{O}$ (Octonions) and $A_4 = \mathbb{S}$ (Sedenions). The concept of the quaternion algebra, $4-$dimensional associative and a non-commutative algebra over $\mathbb{R}$, was discovered by William Rowan Hamilton in October 1843. Two months later, John Graves made a discovery on $8-$dimensional associative and a non-commutative algebra over $\mathbb{R}$ and called as octonions. Quaternions and high order dimension algebras, arise in many areas especially in mathematics, coding theory, physics, robotics, computer science, etc. In recent years, several researchers have studied the quaternions and their generalizations [2, 7–9, 11, 13, 14, 17]. For example, Halıcı investigated the Fibonacci and Lucas quaternions and presented their generating functions, the Binet formulas [9]. She also derived some sums formulas for these quaternions. Keçilioğlu and Akkuş defined the Fibonacci and Lucas octonions and they gave some identities such as Catalan identity, Cassini's identity and d'Ocagne's identity for these octonions [11]. Bilgici et al. proposed the Fibonacci and Lucas sedenions, $16-$dimensional non-associative and non-commutative algebra over $\mathbb{R}$, and then they gave some identities for these sedenions by using the Binet formula [2]. Gül illustrated the $k-$Fibonacci and $k-$Lucas trigintaduonions, $32-$dimensional non-associative and non-commutative algebra over $\mathbb{R}$, and she gave some properties of these trigintaduonions and derive relationships between them [8]. Göcen and Soykan defined the Horadam 2^k-ions, which are the generalization of the some earlier studies, and investigated their properties [7]. Moreover, the authors defined the 2^k-ions $S \in A_k$ as

$$S = \sum_{i=0}^{N-1} a_i e_i = a_0 + \sum_{i=1}^{N-1} a_i e_i, \tag{9}$$

where $N = 2^k$ is the dimension of A_k, e_0 is the unit, $e_1, e_2, \dots, e_{N-1}$ are imaginaries and $a_0, a_1, a_2, \dots, a_{N-1}$ are real numbers. Furthermore, for $S_1, S_2 \in A_k$, the multiplication of two 2^k-ions are

$$S_1 S_2 = \left(\sum_{i=0}^{N-1} a_i e_i \right) \left(\sum_{i=0}^{N-1} b_i e_i \right) = \sum_{i,j=0}^{N-1} a_i b_j (e_i e_j). \tag{10}$$

In this paper, by analogy to the earlier studies, we define a new generalization of the 2^k-ions. The rest of the paper is organized as follows. In section 2 and 3, we define the modified feneralized Fibonacci 2^k-ions and modified feneralized Lucas 2^k-ions, respectively. Besides that, we give generating functions, Binet formulas and some well-known identities for these 2^k-ions. In the last section, we give a concise conclusion.

2 Modified generalized Fibonacci 2^k-ions

In this section, by virtue of the Eq. (3), we define the modified generalized Fibonacci 2^k-ions. By the help of formal power series representation, we give the generating functions for these 2^k-ions. Also, we derive the Binet formula for the modified generalized Fibonacci 2^k-ions with the help of Eq. (7).

Definition 1. *For $n \in \mathbb{N}_0$, the modified generalized Fibonacci 2^k-ions Θ_n is defined by*

$$\Theta_n = \sum_{l=0}^{N-1} Q_{n+l} e_l, \tag{11}$$

where Q_n is the nth modified generalized Fibonacci numbers that is defined in (3).

It is clear from the following Table 1 that the modified generalized Fibonacci 2^k-ions are the generalization of many studies in the literature for the special cases of a, b, c, d and k.

a	***b***	***c***	***d***	***k***	***Modified Generalized Fibonacci*** 2^k-***ions***
1	1	1	1	2	Fibonacci quaternions [10]
a	b	1	1	2	Biperiodic Fibonacci quaternions [17]
k	k	1	1	2	$k-$Fibonacci quaternions [14]
2	2	1	1	2	Pell quaternions [15]
1	1	2	2	2	Jacobsthal quaternions [16]
1	1	1	1	3	Fibonacci octonions [11]
a	b	1	1	3	Biperiodic Fibonacci octonions [22]
k	k	1	1	3	$k-$Fibonacci octonions
2	2	1	1	3	Pell octonions [15]
1	1	2	2	3	Jacobsthal octonions [4]
1	1	1	1	4	Fibonacci sedenions [2]
a	b	1	1	4	Biperiodic Fibonacci sedenions
k	k	1	1	4	$k-$Fibonacci sedenions
2	2	1	1	4	Pell sedenions
1	1	2	2	4	Jacobsthal sedenions

Table 1. The modified generalized Fibonacci 2^k-ions

Theorem 2.1. *The generating function for the modified generalized Fibonacci* 2^k-*ion* Θ_n *is*

$$G(t) = \frac{\Theta_0 + (\Theta_1 - b\Theta_0)\,t + (a-b)R_1(t) + (c-d)R_2(t)}{1 - bt - dt^2}, \tag{12}$$

where

$$R_1(t) = e_0 t f(t) + \sum_{l=1}^{N-1} e_l \left(\frac{f(t) - \sum_{s=1}^{\left\lfloor \frac{l+1}{2} \right\rfloor} Q_{2s-1} t^{2s-1}}{t^{l-1}} \right)$$

$$R_2(t) = \sum_{l=0}^{2} e_l t^{2-l} h(t) + \sum_{l=3}^{N-1} e_l \left(\frac{h(t) - \sum_{s=1}^{\left\lfloor \frac{l-1}{2} \right\rfloor} Q_{2s} t^{2s}}{t^{l-2}} \right)$$

$$f(t) = \frac{t - ct^3}{1 - (ab+d+c)\,t^2 + cdt^4},$$

$$h(t) = \frac{at^2}{1 - (ab+d+c)\,t^2 + cdt^4}.$$

Proof. We use formal power series representation in order to find the generating function of Θ_m. Now we define

$$G(t) = \sum_{m=0}^{\infty} \Theta_m t^m = \Theta_0 + \Theta_1 t + \sum_{m=2}^{\infty} \Theta_m t^m. \tag{13}$$

Note that,

$$btG(t) = \sum_{m=0}^{\infty} b\Theta_m t^{m+1} = \sum_{m=1}^{\infty} b\Theta_{m-1} t^m = bt\Theta_0 + \sum_{m=2}^{\infty} b\Theta_{m-1} t^m \tag{14}$$

and

$$dt^2G(t) = \sum_{m=0}^{\infty} d\Theta_m t^{m+2} = \sum_{m=2}^{\infty} d\Theta_{m-2} t^m. \tag{15}$$

Since Q_n satisfies the recurrence relations $Q_{2m} = aQ_{2m-1} + cQ_{2m-2}$ and $Q_{2m+1} = bQ_{2m} + dQ_{2m-1}$, we obtain

$$\begin{aligned}
&\left(1-bt-dt^2\right)G(t)\\
&\quad= \Theta_0 + (\Theta_1 - b\Theta_0)\,t + \sum_{m=2}^{\infty}\left(\Theta_m - b\Theta_{m-1} - d\Theta_{m-2}\right)t^m\\
&\quad= \Theta_0 + (\Theta_1 - b\Theta_0)\,t\\
&\qquad+ e_0\left((a-b)t\sum_{m=1}^{\infty} Q_{2m-1}t^{2m-1} + (c-d)t^2\sum_{m=1}^{\infty} Q_{2m-2}t^{2m-2}\right)\\
&\qquad+ e_1\left((a-b)\sum_{m=2}^{\infty} Q_{2m-1}t^{2m-1} + (c-d)t\sum_{m=2}^{\infty} Q_{2m-2}t^{2m-2}\right)\\
&\qquad+ e_2\left(\left(\frac{a-b}{t}\right)\sum_{m=2}^{\infty} Q_{2m-1}t^{2m-1} + (c-d)\sum_{m=2}^{\infty} Q_{2m-2}t^{2m-2}\right)\\
&\qquad+ e_3\left(\left(\frac{a-b}{t^2}\right)\sum_{m=3}^{\infty} Q_{2m-1}t^{2m-1} + \left(\frac{c-d}{t}\right)\sum_{m=3}^{\infty} Q_{2m-2}t^{2m-2}\right)\\
&\qquad+ \cdots +\\
&\qquad+ e_{N-1}\left(\left(\frac{a-b}{t^{N-2}}\right)\sum_{m=\left\lfloor\frac{N+2}{2}\right\rfloor}^{\infty} Q_{2m-1}t^{2m-1} + \left(\frac{c-d}{t^{N-3}}\right)\sum_{m=\left\lfloor\frac{N+2}{2}\right\rfloor}^{\infty} Q_{2m-2}t^{2m-2}\right)\\
&\quad= \Theta_0 + (\Theta_1 - b\Theta_0)\,t\\
&\qquad+ \sum_{l=0}^{N-1} e_l\left(\left(\frac{a-b}{t^{l-1}}\right)\sum_{m=\left\lfloor\frac{l+3}{2}\right\rfloor}^{\infty} Q_{2m-1}t^{2m-1} + \left(\frac{c-d}{t^{l-2}}\right)\sum_{m=\left\lfloor\frac{l+3}{2}\right\rfloor}^{\infty} Q_{2m-2}t^{2m-2}\right)\\
&\quad= \Theta_0 + (\Theta_1 - b\Theta_0)\,t\\
&\qquad+ e_0\left((a-b)\,tf(t) + (c-d)\,t^2h(t)\right)\\
&\qquad+ e_1\left((a-b)\left(f(t) - Q_1t\right) + (c-d)\,th(t)\right)\\
&\qquad+ e_2\left(\left(\frac{a-b}{t}\right)\left(f(t) - Q_1t\right) + (c-d)\,h(t)\right)\\
&\qquad+ e_3\left(\left(\frac{a-b}{t^2}\right)\left(f(t) - Q_1t - Q_3t^3\right) + \left(\frac{c-d}{t}\right)\left(h(t) - Q_2t^2\right)\right)\\
&\qquad+ \cdots +\\
&\qquad+ e_{N-1}\left(\left(\frac{a-b}{t^{N-2}}\right)\right)\left(f(t) - Q_1t - Q_3t^3 - Q_5t^5 - \cdots - Q_{N-1-\xi(N)}t^{N-1-\xi(N)}\right)\\
&\qquad+ e_{N-1}\left(\left(\frac{c-d}{t^{N-3}}\right)\right)\left(h(t) - Q_2t^2 - Q_4t^4 - Q_6t^6 - \cdots - Q_{N-3+\xi(N-1)}t^{N-3+\xi(N-1)}\right)\\
&\quad= \Theta_0 + (\Theta_1 - b\Theta_0)\,t + (a-b)R_1(t) + (c-d)R_2(t),
\end{aligned}$$

where

$$
\begin{aligned}
R_1(t) &= e_0 t f(t) + \sum_{l=1}^{N-1} e_l \left(\frac{f(t) - \sum_{s=1}^{\left\lfloor \frac{l+1}{2} \right\rfloor} Q_{2s-1} t^{2s-1}}{t^{l-1}} \right) \\
R_2(t) &= \sum_{l=0}^{2} e_l t^{2-l} h(t) + \sum_{l=3}^{N-1} e_l \left(\frac{h(t) - \sum_{s=1}^{\left\lfloor \frac{l-1}{2} \right\rfloor} Q_{2s} t^{2s}}{t^{l-2}} \right) \\
f(t) &= \sum_{m=1}^{\infty} Q_{2m-1} t^{2m-1} \\
h(t) &= \sum_{m=1}^{\infty} Q_{2m-2} t^{2m-2}.
\end{aligned}
$$

On the other hand, the modified generalized Fibonacci numbers satisfy

$$
\begin{aligned}
Q_{2m-1} &= bQ_{2m-2} + dQ_{2m-3} \\
&= b\left(aQ_{2m-3} + cQ_{2m-4}\right) + dQ_{2m-3} \\
&= \left(ab + d\right) Q_{2m-3} + bcQ_{2m-4} \\
&= \left(ab + d\right) Q_{2m-3} + cQ_{2m-3} - cdQ_{2m-5} \\
&= \left(ab + d + c\right) Q_{2m-3} - cdQ_{2m-5},
\end{aligned}
\tag{16}
$$

and

$$
\begin{aligned}
Q_{2m-2} &= aQ_{2m-3} + cQ_{2m-4} \\
&= a\left(bQ_{2m-4} + dQ_{2m-5}\right) + cQ_{2m-4} \\
&= \left(ab + c\right) Q_{2m-4} + adQ_{2m-5} \\
&= \left(ab + c\right) Q_{2m-4} + dQ_{2m-4} - cdQ_{2m-6} \\
&= \left(ab + d + c\right) Q_{2m-4} - cdQ_{2m-6}.
\end{aligned}
\tag{17}
$$

Using (16) and (17), we obtain

$$
\begin{aligned}
\left(1 - \left(ab + d + c\right) t^2 + cdt^4\right) f(t) =& t + \left(ab + d\right) t^3 - \left(ab + d + c\right) t^3 \\
&+ \sum_{m=3}^{\infty} \left(Q_{2m-1} - \left(ab + d + c\right) Q_{2m-3} + cdQ_{2m-5}\right) t^{2m-1},
\end{aligned}
$$

and

$$
\left(1 - \left(ab + d + c\right) t^2 + cdt^4\right) h(t) = at^2 + \sum_{m=3}^{\infty} \left(Q_{2m-2} - \left(ab + d + c\right) Q_{2m-4} + cdQ_{2m-6}\right) t^{2m-2}.
$$

Rearranging the above expressions, we get

$$
f(t) = \frac{t - ct^3}{1 - \left(ab + d + c\right) t^2 + cdt^4}
$$

and

$$
h(t) = \frac{at^2}{1 - \left(ab + d + c\right) t^2 + cdt^4}.
$$

Therefore, by using $f(t)$, $h(t)$, $R_1(t)$ and $R_2(t)$, we obtain the generating function of Θ_n as:

$$G(t) = \frac{\Theta_0 + (\Theta_1 - b\Theta_0)\, t + (a-b)R_1(t) + (c-d)R_2(t)}{1 - bt - dt^2}. \tag{18}$$

This completes the proof. $\square$

Now, we derive the Binet formula of the modified generalized Fibonacci 2^k-ion by the help of the Binet formula of Q_n.

Theorem 2.2. *For $n \in \mathbb{N}_0$, the Binet formula for the modified generalized Fibonacci 2^k-ion is*

$$\Theta_n = \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \frac{\alpha_{\xi(n)}\alpha^{\lfloor \frac{n}{2} \rfloor}\left(\alpha + d - c\right)^{n - \lfloor \frac{n}{2} \rfloor} - \beta_{\xi(n)}\beta^{\lfloor \frac{n}{2} \rfloor}\left(\beta + d - c\right)^{n - \lfloor \frac{n}{2} \rfloor}}{\alpha - \beta}, \tag{19}$$

where

$$\alpha_{\xi(n)} = \sum_{l=0}^{N-1} \frac{a^{\xi(l+1-\xi(n))}}{(ab)^{\lfloor \frac{l+\xi(n)}{2} \rfloor}} \left(\alpha + d - c\right)^{\lfloor \frac{l+1-\xi(n)}{2} \rfloor} \alpha^{\lfloor \frac{l+\xi(n)}{2} \rfloor} e_l$$

and

$$\beta_{\xi(n)} = \sum_{l=0}^{N-1} \frac{a^{\xi(l+1-\xi(n))}}{(ab)^{\lfloor \frac{l+\xi(n)}{2} \rfloor}} \left(\beta + d - c\right)^{\lfloor \frac{l+1-\xi(n)}{2} \rfloor} \beta^{\lfloor \frac{l+\xi(n)}{2} \rfloor} e_l.$$

Proof. By using the Binet formula of the modified generalized Fibonacci sequence, we can write

$$\begin{aligned}
\Theta_{2n} &= \sum_{l=0}^{N-1} Q_{2n+l} e_l \\
&= e_0 \frac{a}{(ab)^n} \left(\frac{\alpha^n \left(\alpha + d - c\right)^n - \beta^n \left(\beta + d - c\right)^n}{\alpha - \beta} \right) \\
&\quad + e_1 \frac{1}{(ab)^n} \left(\frac{\alpha^n \left(\alpha + d - c\right)^{n+1} - \beta^n \left(\beta + d - c\right)^{n+1}}{\alpha - \beta} \right) \\
&\quad + e_2 \frac{a}{(ab)^{n+1}} \left(\frac{\alpha^{n+1} \left(\alpha + d - c\right)^{n+1} - \beta^{n+1} \left(\beta + d - c\right)^{n+1}}{\alpha - \beta} \right) \\
&\quad + e_3 \frac{1}{(ab)^{n+1}} \left(\frac{\alpha^{n+1} \left(\alpha + d - c\right)^{n+2} - \beta^{n+1} \left(\beta + d - c\right)^{n+2}}{\alpha - \beta} \right) \\
&\quad + \dots \\
&\quad + e_{N-2} \frac{a}{(ab)^{n+\frac{N-2}{2}}} \left(\frac{\alpha^{n+\frac{N-2}{2}} \left(\alpha + d - c\right)^{n+\frac{N-2}{2}} - \beta^{n+\frac{N-2}{2}} \left(\beta + d - c\right)^{n+\frac{N-2}{2}}}{\alpha - \beta} \right) \\
&\quad + e_{N-1} \frac{1}{(ab)^{n+\frac{N-2}{2}}} \left(\frac{\alpha^{n+\frac{N-2}{2}} \left(\alpha + d - c\right)^{n+\frac{N}{2}} - \beta^{n+\frac{N-2}{2}} \left(\beta + d - c\right)^{n+\frac{N}{2}}}{\alpha - \beta} \right)
\end{aligned}$$

$$\begin{aligned}
\Theta_{2n} = \frac{1}{(ab)^n} \frac{\alpha^n \left(\alpha + d - c\right)^n}{\alpha - \beta} \times \Bigg(& e_0 a + e_1(\alpha + d - c) + e_2 \left(\frac{a\alpha(\alpha + d - c)}{ab} \right) \\
& + e_3 \left(\frac{\alpha(\alpha + d - c)^2}{ab} \right) + \dots + e_{N-2} \left(\frac{a\alpha^{\frac{N-2}{2}}(\alpha + d - c)^{\frac{N-2}{2}}}{(ab)^{\frac{N-2}{2}}} \right)
\end{aligned}$$

$$
\begin{aligned}
&+e_{N-1}\left(\frac{\alpha^{\frac{N-2}{2}}(\alpha+d-c)^{\frac{N}{2}}}{(ab)^{\frac{N-2}{2}}}\right)\Bigg)\\
&-\frac{1}{(ab)^{n}}\frac{\beta^{n}(\beta+d-c)^{n}}{\alpha-\beta}\times\Bigg(e_0a+e_1(\beta+d-c)+e_2\left(\frac{a\beta(\beta+d-c)}{ab}\right)\\
&+e_3\left(\frac{\beta(\beta+d-c)^2}{ab}\right)+\cdots+e_{N-2}\left(\frac{a\beta^{\frac{N-2}{2}}(\beta+d-c)^{\frac{N-2}{2}}}{(ab)^{\frac{N-2}{2}}}\right)\\
&+e_{N-1}\left(\frac{\beta^{\frac{N-2}{2}}(\beta+d-c)^{\frac{N}{2}}}{(ab)^{\frac{N-2}{2}}}\right)\Bigg)\\
&=\frac{1}{(ab)^{n}}\frac{\alpha_0\alpha^{n}(\alpha+d-c)^{n}-\beta_0\beta^{n}(\beta+d-c)^{n}}{\alpha-\beta},
\end{aligned}
\tag{20}
$$

where

$$\alpha_0=\sum_{l=0}^{N-1}\frac{a^{\xi(l+1)}}{(ab)^{\lfloor\frac{l}{2}\rfloor}}(\alpha+d-c)^{\lfloor\frac{l+1}{2}\rfloor}\alpha^{\lfloor\frac{l}{2}\rfloor}e_l$$

and

$$\beta_0=\sum_{l=0}^{N-1}\frac{a^{\xi(l+1)}}{(ab)^{\lfloor\frac{l}{2}\rfloor}}(\beta+d-c)^{\lfloor\frac{l+1}{2}\rfloor}\beta^{\lfloor\frac{l}{2}\rfloor}e_l.$$

Similarly, we can obtain

$$\Theta_{2n+1}=\frac{1}{(ab)^{n}}\frac{\alpha_1\alpha^{n}(\alpha+d-c)^{n+1}-\beta_1\beta^{n}(\beta+d-c)^{n+1}}{\alpha-\beta},\tag{21}$$

where

$$\alpha_1=\sum_{l=0}^{N-1}\frac{a^{\xi(l)}}{(ab)^{\lfloor\frac{l+1}{2}\rfloor}}(\alpha+d-c)^{\lfloor\frac{l}{2}\rfloor}\alpha^{\lfloor\frac{l+1}{2}\rfloor}e_l$$

and

$$\beta_1=\sum_{l=0}^{N-1}\frac{a^{\xi(l)}}{(ab)^{\lfloor\frac{l+1}{2}\rfloor}}(\beta+d-c)^{\lfloor\frac{l}{2}\rfloor}\beta^{\lfloor\frac{l+1}{2}\rfloor}e_l.$$

Combining the equations (20) and (21), we get

$$\Theta_n=\frac{1}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\frac{\alpha_{\xi(n)}\alpha^{\lfloor\frac{n}{2}\rfloor}(\alpha+d-c)^{n-\lfloor\frac{n}{2}\rfloor}-\beta_{\xi(n)}\beta^{\lfloor\frac{n}{2}\rfloor}(\beta+d-c)^{n-\lfloor\frac{n}{2}\rfloor}}{\alpha-\beta},$$

where

$$\alpha_{\xi(n)}=\sum_{l=0}^{N-1}\frac{a^{\xi(l+1-\xi(n))}}{(ab)^{\lfloor\frac{l+\xi(n)}{2}\rfloor}}(\alpha+d-c)^{\lfloor\frac{l+1-\xi(n)}{2}\rfloor}\alpha^{\lfloor\frac{l+\xi(n)}{2}\rfloor}e_l$$

and

$$\beta_{\xi(n)}=\sum_{l=0}^{N-1}\frac{a^{\xi(l+1-\xi(n))}}{(ab)^{\lfloor\frac{l+\xi(n)}{2}\rfloor}}(\beta+d-c)^{\lfloor\frac{l+1-\xi(n)}{2}\rfloor}\beta^{\lfloor\frac{l+\xi(n)}{2}\rfloor}e_l.\qquad\square$$

In the following theorem we derive the Catalan's identity with the help of the Binet formula of Q_n. Furthermore, we give the Cassini's identity which is the special case of the Catalan's identity for $r=1$.

Theorem 2.3 (Catalan's identity). *For $n, r \in \mathbb{N}_0$ and $r \leq n$, we have the identity*

$$\begin{aligned}
&\Theta_{2(n+r)+\xi(i)}\Theta_{2(n-r)+\xi(i)} - \Theta_{2n+\xi(i)}^2 \\
&\quad= \frac{(-c)^{\xi(i)}}{(ab)^{2r}(\alpha-\beta)^2} \\
&\quad\times\left[\alpha_{\xi(i)}\beta_{\xi(i)}\left((ab)^{2r+\xi(i)}(cd)^n - (ab)^{2r+\xi(i)}(cd)^n\left(\frac{\alpha+d}{\beta+d}\right)^r\right)\right. \\
&\quad\left.+\beta_{\xi(i)}\alpha_{\xi(i)}\left((ab)^{2r+\xi(i)}(cd)^n - (ab)^{2r+\xi(i)}(cd)^n\left(\frac{\beta+d}{\alpha+d}\right)^r\right)\right],
\end{aligned}$$

where $\alpha_{\xi(i)}$ and $\beta_{\xi(i)}$ are defined in Theorem 2.2 and $i \in \{0,1\}$.

Proof. By using the Binet formula of the modified generalized Fibonacci 2^k-ion, for $i = 0$, we get

$$\begin{aligned}
&\Theta_{2(n+r)}\Theta_{2(n-r)} - \Theta_{2n}^2 \\
&=\left(\frac{1}{(ab)^{n+r}}\frac{\alpha_0\alpha^{n+r}(\alpha+d-c)^{n+r} - \beta_0\beta^{n+r}(\beta+d-c)^{n+r}}{\alpha-\beta}\right) \\
&\times\left(\frac{1}{(ab)^{n-r}}\frac{\alpha_0\alpha^{n-r}(\alpha+d-c)^{n-r} - \beta_0\beta^{n-r}(\beta+d-c)^{n-r}}{\alpha-\beta}\right) \\
&-\left(\frac{1}{(ab)^{n}}\frac{\alpha_0\alpha^{n}(\alpha+d-c)^{n} - \beta_0\beta^{n}(\beta+d-c)^{n}}{\alpha-\beta}\right)^2 \\
&=\frac{1}{(ab)^{2n}(\alpha-\beta)^2}\Big(\alpha_0\beta_0\Big(\alpha^n\beta^n(\alpha+d-c)^n(\beta+d-c)^n \\
&-\alpha^{n+r}\beta^{n-r}(\alpha+d-c)^{n+r}(\beta+d-c)^{n-r}\Big) \\
&+\beta_0\alpha_0\Big(\alpha^n\beta^n(\alpha+d-c)^n(\beta+d-c)^n \\
&-\alpha^{n-r}\beta^{n+r}(\alpha+d-c)^{n-r}(\beta+d-c)^{n+r}\Big)\Big) \\
&=\frac{1}{(ab)^{2r}(\alpha-\beta)^2}\left(\alpha_0\beta_0\left((ab)^{2r}(cd)^n - (ab)^{2r}(cd)^n\left(\frac{\alpha+d}{\beta+d}\right)^r\right)\right. \\
&\left.+\beta_0\alpha_0\left((ab)^{2r}(cd)^n - (ab)^{2r}(cd)^n\left(\frac{\beta+d}{\alpha+d}\right)^r\right)\right).
\end{aligned} \tag{22}$$

Similarly, for $i = 1$, we get

$$\begin{aligned}
&\Theta_{2(n+r)+1}\Theta_{2(n-r)+1} - \Theta_{2n+1}^2 \\
&\quad= -\frac{c}{(ab)^{2r}(\alpha-\beta)^2}\left(\alpha_1\beta_1\left((ab)^{2r+1}(cd)^n - (ab)^{2r+1}(cd)^n\left(\frac{\alpha+d}{\beta+d}\right)^r\right)\right. \\
&\quad\left.+\beta_1\alpha_1\left((ab)^{2r+1}(cd)^n - (ab)^{2r+1}(cd)^n\left(\frac{\beta+d}{\alpha+d}\right)^r\right)\right).
\end{aligned} \tag{23}$$

By combining the equations (22) and (23), we obtain

$$\begin{aligned}
&\Theta_{2(n+r)+\xi(i)}\Theta_{2(n-r)+\xi(i)} - \Theta^2_{2n+\xi(i)} \\
&\quad = \frac{(-c)^{\xi(i)}}{(ab)^{2r}(\alpha-\beta)^2} \\
&\qquad \times\left[\alpha_{\xi(i)}\beta_{\xi(i)}\left((ab)^{2r+\xi(i)}(cd)^n - (ab)^{2r+\xi(i)}(cd)^n\left(\frac{\alpha+d}{\beta+d}\right)^r\right)\right. \\
&\qquad \left.+\beta_{\xi(i)}\alpha_{\xi(i)}\left((ab)^{2r+\xi(i)}(cd)^n - (ab)^{2r+\xi(i)}(cd)^n\left(\frac{\beta+d}{\alpha+d}\right)^r\right)\right],
\end{aligned}$$

where $\alpha_{\xi(i)}$ and $\beta_{\xi(i)}$ are defined in Theorem 2.2 and $i \in \{0,1\}$. □

Corollary 2.3.1 (Cassini's identity). *For $n \in \mathbb{N}_0$, we have the identity*

$$\begin{aligned}
&\Theta_{2(n+1)+\xi(i)}\Theta_{2(n-1)+\xi(i)} - \Theta^2_{2n+\xi(i)} \\
&\quad = \frac{(-c)^{\xi(i)}}{(ab)^{2}(\alpha-\beta)^2} \\
&\qquad \times\left[\alpha_{\xi(i)}\beta_{\xi(i)}\left((ab)^{2+\xi(i)}(cd)^n - (ab)^{2+\xi(i)}(cd)^n\left(\frac{\alpha+d}{\beta+d}\right)\right)\right. \\
&\qquad \left.+\beta_{\xi(i)}\alpha_{\xi(i)}\left((ab)^{2+\xi(i)}(cd)^n - (ab)^{2+\xi(i)}(cd)^n\left(\frac{\beta+d}{\alpha+d}\right)\right)\right],
\end{aligned}$$

where $\alpha_{\xi(i)}$ and $\beta_{\xi(i)}$ are defined in Theorem 2.2 and $i \in \{0,1\}$.

3 Modified generalized Lucas 2^k-ions

In this section, we define the modified generalized Lucas 2^k-ion ϑ_n. We give the generating function, the Binet formula and some important identities for this 2^k-ion. The theorems and results in this section can be proven similar to the results in Section 2. Hence, we omit the proofs.

Definition 2. *For $n \in \mathbb{N}_0$, the modified generalized Lucas 2^k-ion ϑ_n is defined by*

$$\vartheta_n = \sum_{l=0}^{N-1} U_{n+l}e_l, \tag{24}$$

where U_n is the modified generalized Lucas numbers that is defined in (4).

It is clear from the following Table 2 that the modified generalized Lucas 2^k-ions are the generalization of many studies in the literature for the special cases of a, b, c, d and k.

a	***b***	***c***	***d***	***k***	***Modified Generalized Lucas 2^k-ions***
1	1	1	1	2	Lucas quaternions [10]
a	b	1	1	2	Biperiodic Lucas quaternions [18]
k	k	1	1	2	$k-$Lucas quaternions [14]
1	1	1	1	3	Lucas octonions [11]
a	b	1	1	3	Biperiodic Lucas octonions [23]
k	k	1	1	3	$k-$Lucas octonions
1	1	1	1	4	Lucas sedenions [2]
a	b	1	1	4	Biperiodic Lucas sedenions
k	k	1	1	4	$k-$Lucas sedenions

Table 2. The modified generalized Lucas 2^k-ions

Theorem 3.1. *The generating function for the modified generalized Lucas 2^k-ion ϑ_n is*

$$L(t)=\frac{\vartheta_0+(\vartheta_1-a\vartheta_0)\,t+(b-a)R_1(t)+(d-c)R_2(t)}{1-at-ct^2}, \tag{25}$$

where

$$R_1(t)=e_0tf(t)+\sum_{l=1}^{N-1}e_l\left(\frac{f(t)-\sum_{s=1}^{\lfloor\frac{l+1}{2}\rfloor}U_{2s-1}t^{2s-1}}{t^{l-1}}\right)$$

$$R_2(t)=\sum_{l=0}^{2}e_lt^{2-l}h(t)-\sum_{l=1}^{2}e_lt^{2-l}U_0+\sum_{l=3}^{N-1}e_l\left(\frac{h(t)-\sum_{s=0}^{\lfloor\frac{l-1}{2}\rfloor}U_{2s}t^{2s}}{t^{l-2}}\right)$$

$$f(t)=\frac{at+at^3}{1-(ab+d+c)\,t^2+cdt^4},$$

$$h(t)=\frac{\left(\frac{d+1}{d}\right)+(ab+d+1)t^2-(ab+d+c)\left(\frac{d+1}{d}\right)t^2}{1-(ab+d+c)\,t^2+cdt^4}.$$

Proof. Proof can be made similarly to Theorem (2.1). ☐

Theorem 3.2. *For $n\in\mathbb{N}_0$, the Binet formula for the modified generalized Lucas 2^k-ion is*

$$\vartheta_n = \frac{1}{(ab)^{\lfloor\frac{n-1}{2}\rfloor}}\left(\frac{\alpha^{\star}_{\xi(n)}(\alpha+d+1)\alpha^{\lfloor\frac{n-1}{2}\rfloor}(\alpha+d-c)^{\lfloor\frac{n}{2}\rfloor}}{\alpha-\beta}-\frac{\beta^{\star}_{\xi(n)}(\beta+d+1)\beta^{\lfloor\frac{n-1}{2}\rfloor}(\beta+d-c)^{\lfloor\frac{n}{2}\rfloor}}{\alpha-\beta}\right), \tag{26}$$

where

$$\alpha^{\star}_{\xi(n)}=\sum_{l=0}^{N-1}\frac{a^{\xi(l+\xi(n))}}{(ab)^{\lfloor\frac{l+1-\xi(n)}{2}\rfloor}}(\alpha+d-c)^{\lfloor\frac{l+\xi(n)}{2}\rfloor}\alpha^{\lfloor\frac{l+1-\xi(n)}{2}\rfloor}e_l$$

and

$$\beta^{\star}_{\xi(n)}=\sum_{l=0}^{N-1}\frac{a^{\xi(l+\xi(n))}}{(ab)^{\lfloor\frac{l+1-\xi(n)}{2}\rfloor}}(\beta+d-c)^{\lfloor\frac{l+\xi(n)}{2}\rfloor}\beta^{\lfloor\frac{l+1-\xi(n)}{2}\rfloor}e_l.$$

Proof. The proof can be made similarly to Theorem 2.2. ☐

In the following theorem we derive the Catalan's identity with the help of the Binet formula of U_n. Furthermore, we give the Cassini's identity which is the special case of the Catalan's identity for $r = 1$.

Theorem 3.3 (Catalan's identity). *For $n, r \in \mathbb{N}_0$ and $r \leq n$, we have the identity*

$$\begin{aligned}\vartheta_{2(n+r)+\xi(i)}\vartheta_{2(n-r)+\xi(i)} - \vartheta^2_{2n+\xi(i)} &= \frac{(-c)^{1-\xi(i)}(\alpha+d+1)(\beta+d+1)}{(ab)^{2r}(\alpha-\beta)^2} \\ \times \Bigg[\alpha^\star_{\xi(i)}\beta^\star_{\xi(i)}&\left((ab)^{2r+1-\xi(i)}(cd)^{n-1+\xi(i)} - (ab)^{2r+1-\xi(i)}(cd)^{n-1+\xi(i)}\left(\frac{\alpha+d}{\beta+d}\right)^r\right) \\ + \beta^\star_{\xi(i)}\alpha^\star_{\xi(i)}&\left((ab)^{2r+1-\xi(i)}(cd)^{n-1+\xi(i)} - (ab)^{2r+1-\xi(i)}(cd)^{n-1+\xi(i)}\left(\frac{\beta+d}{\alpha+d}\right)^r\right)\Bigg],\end{aligned}$$

where $\alpha^\star_{\xi(i)}$ and $\beta^\star_{\xi(i)}$ are defined in Theorem 3.2 and $i \in \{0, 1\}$.

Proof. The proof can be made similarly to Theorem 2.3. □

Corollary 3.3.1 (Cassini's identity). *For $n \in \mathbb{N}_0$, we have the identity*

$$\begin{aligned}\vartheta_{2(n+1)+\xi(i)}\vartheta_{2(n-1)+\xi(i)} - \vartheta^2_{2n+\xi(i)} &= \frac{(-c)^{1-\xi(i)}(\alpha+d+1)(\beta+d+1)}{(ab)^{2}(\alpha-\beta)^2} \\ \times \Bigg[\alpha^\star_{\xi(i)}\beta^\star_{\xi(i)}&\left((ab)^{3-\xi(i)}(cd)^{n-1+\xi(i)} - (ab)^{3-\xi(i)}(cd)^{n-1+\xi(i)}\left(\frac{\alpha+d}{\beta+d}\right)\right) \\ + \beta^\star_{\xi(i)}\alpha^\star_{\xi(i)}&\left((ab)^{3-\xi(i)}(cd)^{n-1+\xi(i)} - (ab)^{3-\xi(i)}(cd)^{n-1+\xi(i)}\left(\frac{\beta+d}{\alpha+d}\right)\right)\Bigg],\end{aligned}$$

where $\alpha^\star_{\xi(i)}$ and $\beta^\star_{\xi(i)}$ are defined in Theorem 3.2 and $i \in \{0, 1\}$.

Theorem 3.4. *Let $n \geq 1$ be integer. Then the modified generalized Lucas 2^k-ion satisfies the relation*

$$\vartheta_n = \Theta_{n-1} + \Theta_{n+1}. \tag{27}$$

Proof. By considering the identity $U_n = Q_{n-1} + Q_{n+1}$, which is given by [1, Theorem 20], we can easily obtain the desired result. □

4 Conclusion

In this paper, we define the modified generalized Fibonacci and modified generalized Lucas 2^k-ions. Moreover, we give the Catalan's identity and Cassini's identity. Since our study both generalization of several studies in the literature and includes some new results, it contributes to the literature by providing essential information on the generalization of the 2^k-ions.

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