



## New exact solutions and numerical approximations of the generalized KdV equation

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### Abstract

This paper is devoted to create new exact and numerical solutions of the generalized Korteweg-de Vries (GKdV) equation with ansatz method and Galerkin finite element method based on cubic B-splines over finite elements. Propagation of single solitary wave is investigated to show the efficiency and applicability of the proposed methods. The performance of the numerical algorithm is proved by computing  $L_2$  and  $L_\infty$  error norms. Also, three invariants  $I_1, I_2$ , and  $I_3$  have been calculated to determine the conservation properties of the presented algorithm. The obtained numerical solutions are compared with some earlier studies for similar parameters. This comparison clearly shows that the obtained results are better than some earlier results and they are found to be in good agreement with exact solutions. Additionally, a linear stability analysis based on Von Neumann's theory is surveyed and indicated that our method is unconditionally stable.

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**Keywords.** Generalized Korteweg-de Vries equation, Finite element method, Ansatz method, Galerkin, Cubic B-spline, Soliton.

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### 1. INTRODUCTION

Over many years, most of the difficultiest and interesting natural phenomena being analyzed by mathematicians, physicists, engineers and other scientists are nonlinear in nature. These natural phenomena can be sensitively modeled by several nonlinear partial differential equations (PDEs). The investigation of nonlinear PDE is a main field of mathematics which is especially associated with pure, applied and computational mathematics. In the last few decades, considerable progress has been made in understanding the integrability and non-integrability of nonlinear PDEs [51]. In order to better understand these nonlinear PDEs, it is significant to find out their exact solutions. The solitary wave ansatz method ensures an effective and direct algebraic method for solving nonlinear equations. The method was first suggested by Biswas

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[5] and Triki et al. [56] are especially remarkable in its power and practicability [25]. Also the solitary wave ansatz method [6, 7, 62] is rather intuitional and holds notable properties that make it useful for the determination of single soliton solutions for an extensive class of nonlinear evolution equations with constant and varying coefficients in a direct method. This method is at times named as the trial solution method that is largely governed to solve ordinary differential equations and it is not used to create multiple soliton solutions for integrable equations [30, 56]. But exact solutions of PDE's are widely not gettable, in particular when the nonlinear terms are comprised. In so far as only limited classes of these equations are solved by analytical means therefore numerical solutions of these nonlinear partial differential equations are very practicable to examine physical phenomena. Most known models of such equations involving traveling waves are for instance the nonlinear Korteweg-de Vries (KdV) equation, regularized long wave (RLW) equation, equal width wave (EW) equation and so on. In 1895, two scientists Korteweg and de Vries [37] formulated the most known equation, KdV equation,

$$U_t + \varepsilon U U_x + \mu U_{xxx} = 0, \quad (1.1)$$

for the propagation of waves in one dimension on the surface of water. The KdV equation is a nonlinear PDE of third order. The equation states a balance between dispersion form from its third derivative term  $U_{xxx}$  and the shock forming tendency of its nonlinear term  $U U_x$ . KdV equation has been found to define a great number of physical phenomena such as waves in anharmonic crystals [63], waves in bubble liquid mixtures, ion acoustic wave and magneto-hydrodynamic, waves in a warm plasma as well as shallow water waves [50]. The fundamental feature of the equation is that the speed of solitary wave is related to the magnitude of the solitary wave. Also, another particular property of KdV equation is that solutions, may exhibit solitary wave solutions noted as solitons, which keep their original size, shape and velocity after interaction of solitons [36]. The theory of solitons is a significant field in the areas of physics and mathematics. They occur in biochemistry, nonlinear optics, mathematical biosciences, fluid dynamics, plasma physics, nuclear physics, geophysics and a great many [5]. The KdV equation is a completely integrable Hamiltonian system which can be solved explicitly. This means that it portrays  $N$ -soliton solutions and an infinite number of conserved densities. Some analytical solutions of the KdV equation are invented and their existence and uniqueness have been examined for a certain class of initial functions by Gardner et al. [14, 19, 48]. Such as, the KdV equation is solved analytically by Adomian decomposition method which provides series solutions [59]. The extended mapping transformation method is used to obtain some new exact solutions of a variable-coefficient KdV equation arising in arterial mechanics by [40]. Inan [26] implemented a generalized tanh function method for approximating the solution of the coupled KdV equation and KdV equation. Also, in general, usefulness of these solutions is limited. Therefore, numerical solutions of KdV equation are needful for several boundary and initial conditions to model many physical cases. KdV equation was first solved numerically by Zabusky and Kruskal using finite difference method [64]. There have been various methods to numerically



solve the KdV equation, for instance finite difference method [22, 23, 46, 47, 58], finite element method [4, 11, 15, 20, 21, 24, 28, 29, 53, 54, 55], pseudospectral method [16], variational iteration method [27], the modified Bernstein polynomials [13], meshless method [12, 17] and heat balance integral method [38] etc. have been proposed for numerical treatment of the KdV equation. Actually KdV equation is a special status of GKdV equation given by

$$U_t + \varepsilon U^p U_x + \mu U_{xxx} = 0, \quad (1.2)$$

which has need for the boundary conditions  $\frac{\partial U}{\partial x} \rightarrow 0$  as  $|x| \rightarrow 0$  and where  $\varepsilon, \mu$  are positive parameters and the subscripts  $x$  and  $t$  symbolise spatial and time differentiation, respectively. Numerical solution of the Eq.(1.2) is achieved with boundary conditions taken from

$$\begin{aligned} U(a, t) = 0, \quad U(b, t) = 0, \\ U_x(a, t) = 0, \quad U_x(b, t) = 0, \\ U_{xx}(a, t) = 0, \quad U_{xx}(b, t) = 0, \quad t > 0 \end{aligned} \quad (1.3)$$

and an initial condition

$$U(x, 0) = f(x), \quad a \leq x \leq b. \quad (1.4)$$

We presume that the solution to the Eq.(1.2) is dispensable outside the region  $[a, b]$ . GKdV equation has received much less attention, presumably because of its higher nonlinearity for  $p > 2$ . Solitary waves are explode for  $p = 4$  [3]. The symmetry group was calculated for the equation and several classes of solutions were obtained in [9]. Liu and Yi [42] developed and analyzed a Hamiltonian preserving DG method for solving the generalized KdV equation. The initial value problem of a kind of GKdV equations are considered by using Sobolev space theory and finite element method by Lai et al. [39]. Alvarado and Omel'yanov [18] create a finite differences scheme to simulate the solution of the Cauchy problem and present some numerical results for the problem of the solitary waves interaction. A class of fully discrete scheme for the generalized Korteweg-de Vries equation in a bounded domain  $(0, L)$  has studied by Sep'ulveda and Villagr'an [52]. Collocation finite element method based on quintic B-spline functions is applied to the generalized KdV equation by Ak et al. [3]. Solitary wave solution for the GKdV equation by using ADM has been obtained by Ismail et al. [31]. By applying the multiplier method Bruzon et al. [10] heve obtained a complete classification of low-order local conservation laws for a generalized seventh-order KdV equation depending on seven arbitrary nonzero parameters. The exhaustive group classification of a class of variable coefficient generalized KdV equations has been presented and Lie symmetries have been used for solving an initial and boundary value problem by Vaneeva et al [57]. The explicit solutions to a generalized Korteweg-de Vries equation with initial condition have been calculated by using the Adomian decomposition method by Kaya and Aassila [35].

The another special case of the GKdV equation is the modified Korteweg-de Vries (MKdV) equation for  $p = 2$ . Like the KdV equation, in recent years, various numerical



methods have been improved for the solution of the MKdV equation. Kaya [34], was used the Adomian decomposition method to obtain the higher order modified Korteweg de-Vries equation with initial condition. MKdV equation has been solved by using Galerkin's method with quadratic B-spline finite elements by Biswas et al. [8]. Raslan and Baghdady [44, 45], showed the accuracy and stability of the difference solution of the MKdV equation and they obtained the numerical aspects of the dynamics of shallow water waves along lakes' shores and beaches modeled by the MKdV equation. A new variety of  $(3 + 1)$ -dimensional MKdV equations and multiple soliton solutions for each new equation were established by Wazwaz [60, 61]. A lumped Galerkin and Petrov Galerkin methods were applied to the MKdV equation by Ak et al. [1, 2]. Numerical solutions of the MKdV equation have been obtained both subdomain finite element method using quartic B-splines and collocation finite element method using septic B-splines by Karakoc [32, 33].

The objective of the current study is to offer new exact and numerical schemes to solve the nonlinear third-order GKdV equation, based on ansatz method and cubic B-spline Galerkin method. The main contents of the paper is the following: in section 2 we apply the solitary wave ansatz method to find the exact bright and singular soliton solutions of the GKdV equation. Some graphical illustrations of the obtained solutions for the GKdV equation is given in section 3. Galerkin finite element method has been applied to the equation in section 4. Section 5 contains a linear stability analysis of the scheme followed by section 6 which contains analyzing of the motion of single solitary wave with different initial and boundary conditions. The acquired numerical results are given both in tabular and graphical form and the computed results are also hold a candle to some of those available in the literature. The last section is a brief conclusion.

## 2. NEW EXACT SOLUTIONS OF THE GKdV EQUATION

In this section, the exact solutions of the generalized KdV equation is constructed using ansatz method. We can write the GKdV as follows,

$$D_t U + \epsilon U^p D_x U + \mu D_{xxx} U = 0. \quad (2.1)$$

On employing the wave transformation

$$U(x, t) = f(\xi), \quad \xi = k(x - ct - x_0) \quad (2.2)$$

we get

$$-kcf' + \epsilon k(f^p)f' + \mu k^3 U''' = 0 \quad (2.3)$$

integrating the above equation with respect to  $\xi$  and assuming the constant zero, get

$$-kcf + \frac{\epsilon k}{p+1}(f^{p+1}) + \mu k^3 U'' = 0, \quad (2.4)$$

to solve this equation we need choose the values of  $p = 1, 2, 3$ .



### 2.1. Exact solutions of the GKdV equation using Bright soliton solution.

Let

$$f(\xi) = A \operatorname{sech}^n(\xi), \quad \xi = k(x - ct - x_0). \quad (2.5)$$

From equality (2.5) we get

$$f''(\xi) = An \operatorname{sech}^n(\xi)[-1 + (1+n)\tanh^2(\xi)]. \quad (2.6)$$

If we put  $p = 1$  in Eq.(2.4) and substituting equalities (2.5) and (2.6) into Eq.(2.4) and equating the powers  $n + 2 = 2n$  gives  $n = 2$ . So we derive a system of algebraic equations as follows:

$$\begin{aligned} -Ack + 4Ak^3\mu &= 0, \\ \frac{1}{2}A^2k\epsilon - 6Ak^3\mu &= 0. \end{aligned}$$

When we solve the above system, we obtain the following case of solutions as follow:

$$A = \frac{3c}{\epsilon}, \quad k = \mp \frac{\sqrt{c}}{2\sqrt{\mu}}.$$

Thus for  $p = 1$ , the bright soliton solution is formed as

$$u_{1,2}(x, t) = \frac{3c}{\epsilon} \operatorname{sech}^2(k(x - ct - x_0)). \quad (2.7)$$

If we take  $p = 2$  in Eq.(2.4) and substituting equalities (2.5) and (2.6) into Eq.(2.4) and equating the powers  $n + 2 = 3n$ ,  $n$  is obtained as 1. Thus, we generate a system of algebraic equations as follows:

$$\begin{aligned} -Ack + k^3A\mu &= 0, \\ \frac{1}{3}kA^3\epsilon - 2k^3A\mu &= 0. \end{aligned}$$

Solving the above system, we get the following cases of solutions as follow:

2.1.1. **Case1.** If  $A = -\frac{\sqrt{6}\sqrt{c}}{\sqrt{\epsilon}}$ , and  $k = \mp \frac{\sqrt{c}}{\sqrt{\mu}}$ ,

$$u_{1,2}(x, t) = -\frac{\sqrt{6}\sqrt{c}}{\sqrt{\epsilon}} \operatorname{sech}(k(x - ct - x_0)). \quad (2.8)$$

are procured.

2.1.2. **Case2.** If  $A = \frac{\sqrt{6}\sqrt{c}}{\sqrt{\epsilon}}$  and  $k = \mp \frac{\sqrt{c}}{\sqrt{\mu}}$

$$u_{3,4}(x, t) = \frac{\sqrt{6}\sqrt{c}}{\sqrt{\epsilon}} \operatorname{sech}(k(x - ct - x_0)). \quad (2.9)$$

are attained. If we write  $p = 3$  in Eq.(2.4) and substituting equalities (2.5) and (2.6) into Eq.(2.4) and equating the powers  $n + 2 = 4n$ ,  $n$  is got as  $\frac{2}{3}$ , where the  $n$  is not a positive integer so we must take  $f(\xi) = g^{\frac{2}{3}}(\xi)$ . Then we can rewrite Eq.(2.4) as

$$-ckg^2 + \frac{1}{4}k\epsilon g^4 - \frac{2}{9}k^3\mu g'^2 + \frac{2}{3}k^3\mu g g'' = 0. \quad (2.10)$$



Substituting  $g(\xi) = A \operatorname{sech}^n(\xi)$  and its derivatives into Eq.(2.10) and equating the powers  $2n + 2 = 4n$ , we find  $n$  as 1. Hence we create a system of algebraic equations as follows:

$$\begin{aligned} -A^2kc + \frac{4}{9}k^3A^2\mu &= 0, \\ \frac{1}{4}kA^4\epsilon - \frac{10}{9}k^3A^2\mu &= 0. \end{aligned}$$

Solving the above system, we acquire the following cases of solutions as follow:

2.1.3. **Case1.** If  $A = -\frac{\sqrt{10}\sqrt{c}}{\sqrt{\epsilon}}$  and  $k = \mp \frac{3\sqrt{c}}{2\sqrt{\mu}}$

$$u_{1,2}(x, t) = \left[-\frac{\sqrt{10}\sqrt{c}}{\sqrt{\epsilon}} \operatorname{sech}(k(x - ct - x_0))\right]^{\frac{2}{3}} \tag{2.11}$$

are found.

2.1.4. **Case2.** If  $A = \frac{\sqrt{10}\sqrt{c}}{\sqrt{\epsilon}}$  and  $k = \mp \frac{3\sqrt{c}}{2\sqrt{\mu}}$

$$u_{3,4}(x, t) = \left[\frac{\sqrt{10}\sqrt{c}}{\sqrt{\epsilon}} \operatorname{sech}(k(x - ct - x_0))\right]^{\frac{2}{3}}. \tag{2.12}$$

are derived.

**2.2. Exact solutions of the GKdV equation using Singular soliton solution.**

Let

$$f(\xi) = A \operatorname{csch}^n(\xi), \quad \xi = k(x - ct - x_0). \tag{2.13}$$

From (2.5) we find

$$f''(\xi) = An \operatorname{csch}^n(\xi)[-1 + (1 + n)\operatorname{coth}^2(\xi)]. \tag{2.14}$$

If we take  $p = 1$  in Eq.(2.4) and substituting equalities (2.13) and (2.14) into Eq.(2.4) and equating the powers  $n + 2 = 2n$  gives  $n = 2$ . Consequently we create a system of algebraic equations as follows:

$$\begin{aligned} -Ack + 4Ak^3\mu &= 0, \\ \frac{1}{2}A^2k\epsilon + 6Ak^3\mu &= 0. \end{aligned}$$

When we solve the above system, we obtain the following case of solutions as follow:

$$A = -\frac{3c}{\epsilon}, \quad k = \mp \frac{\sqrt{c}}{2\sqrt{\mu}}.$$

Thus for  $p = 1$ , the singular soliton solution is formed as

$$u_{1,2}(x, t) = \frac{3c}{\epsilon} \operatorname{csch}^2(k(x - ct - x_0)). \tag{2.15}$$

If we take  $p = 2$  in Eq.(2.4) and substituting equalities (2.13) and (2.14) into Eq.(2.4) and equating the powers  $n + 2 = 3n$ ,  $n$  is obtained as 1. Thus, we invent a system of algebraic equations as follows:

$$\begin{aligned} -Ack + k^3A\mu &= 0, \\ \frac{1}{3}kA^3\epsilon + 2k^3A\mu &= 0. \end{aligned}$$

Solving the above system, we get the following cases of solutions as follow:



2.2.1. **Case1.** If  $A = -\frac{i\sqrt{6}\sqrt{c}}{\sqrt{\epsilon}}$  and  $k = \mp\frac{\sqrt{c}}{\sqrt{\mu}}$

$$u_{1,2}(x, t) = -\frac{i\sqrt{6}\sqrt{c}}{\sqrt{\epsilon}} \operatorname{csch}(k(x - ct - x_0)). \quad (2.16)$$

are procured.

2.2.2. **Case2.** If  $A = \frac{i\sqrt{6}\sqrt{c}}{\sqrt{\epsilon}}$  and  $k = \mp\frac{\sqrt{c}}{\sqrt{\mu}}$

$$u_{3,4}(x, t) = \frac{i\sqrt{6}\sqrt{c}}{\sqrt{\epsilon}} \operatorname{csch}(k(x - ct - x_0)). \quad (2.17)$$

are attained. If we write  $p = 3$  in Eq.(2.4) and substituting  $g(\xi) = A \operatorname{csch}^n(\xi)$  and its derivatives into Eq.(2.10) and equating the powers  $2n + 2 = 4n$ ,  $n$  is got as 1. Hence we create a system of algebraic equations as follows:

$$\begin{aligned} -A^2kc + \frac{4}{9}k^3A^2\mu &= 0, \\ \frac{1}{4}kA^4\epsilon + \frac{10}{9}k^3A^2\mu &= 0. \end{aligned}$$

Solving the above system, we acquire the following cases of solutions as follow:

2.2.3. **Case1.** If  $A = -\frac{i\sqrt{10}\sqrt{c}}{\sqrt{\epsilon}}$  and  $k = \mp\frac{3\sqrt{c}}{2\sqrt{\mu}}$ .

$$u_{1,2}(x, t) = \left[-\frac{i\sqrt{10}\sqrt{c}}{\sqrt{\epsilon}} \operatorname{csch}(k(x - ct - x_0))\right]^{\frac{2}{3}}. \quad (2.18)$$

are found.

2.2.4. **Case2.** If  $A = \frac{i\sqrt{10}\sqrt{c}}{\sqrt{\epsilon}}$  and  $k = \mp\frac{3\sqrt{c}}{2\sqrt{\mu}}$ .

$$u_{3,4}(x, t) = \left[\frac{i\sqrt{10}\sqrt{c}}{\sqrt{\epsilon}} \operatorname{csch}(k(x - ct - x_0))\right]^{\frac{2}{3}}. \quad (2.19)$$

are derived.

### 3. SOME GRAPHICAL ILLUSTRATIONS

We depict in this section some graphical illustrations of the obtained solutions for the GKdV equation, both the two and three dimensional plots for the solutions are given.

#### 4. COMPUTER IMPLEMENTATION AND CONSTRUCTION OF THE NUMERICAL METHOD

Let us take care of the solution domain is limited to a finite interval  $a \leq x \leq b$  and separate the interval  $[a, b]$  into  $N$  finite elements of equal length  $h = \frac{b-a}{N} = (x_{m+1} - x_m)$  by knots  $x_m$  such that  $a = x_0 < x_1 < \dots < x_N = b$ . We presume that  $\phi_m(x)$  are these cubic B-splines with knots at  $x_m$ . Prenter [43] expressed following



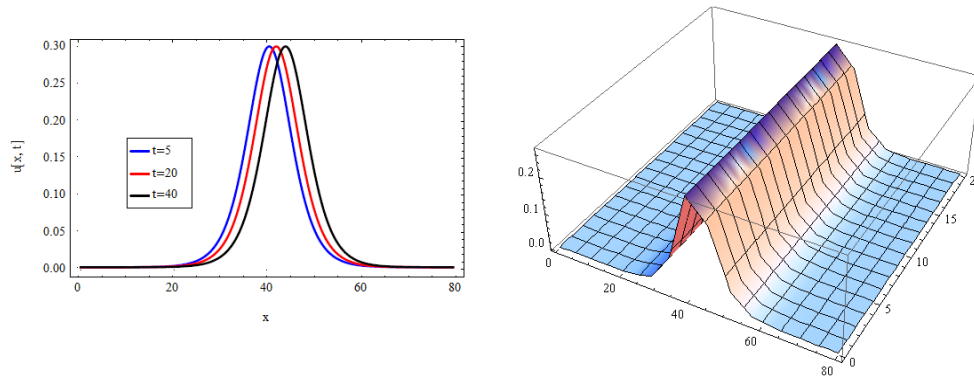


FIGURE 1. Graph of equation (2.7).

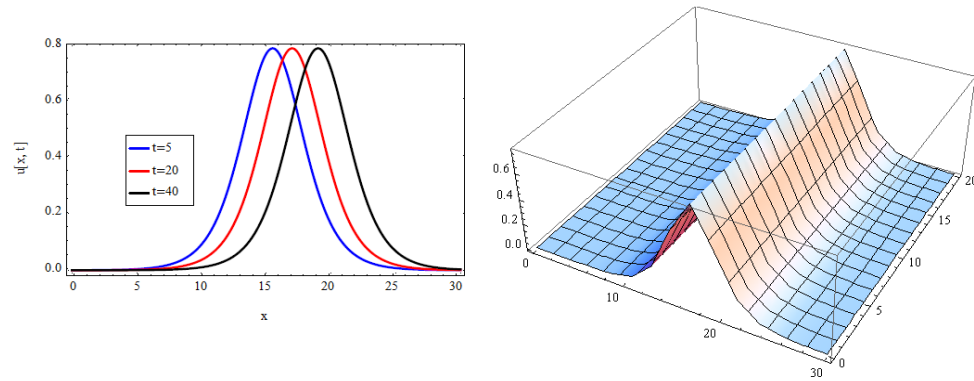


FIGURE 2. Graph of equation (2.8).

cubic B-spline functions  $\phi_m(x)$ , ( $m = -1(1) N + 1$ ), at the points  $x_m$  which generate a basis over the interval  $[a, b]$  by

$$\phi_m(x) = \frac{1}{h^3} \begin{cases} (x - x_{m-2})^3, & x \in [x_{m-2}, x_{m-1}), \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3, & x \in [x_{m-1}, x_m), \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3, & x \in [x_m, x_{m+1}), \\ (x_{m+2} - x)^3, & x \in [x_{m+1}, x_{m+2}), \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

We seek out the global approximation  $U_N(x, t)$  to the solution  $U(x, t)$  which use these splines as the trial functions

$$U_N(x, t) = \sum_{j=-1}^{N+1} \phi_j(x)\delta_j(t), \tag{4.2}$$





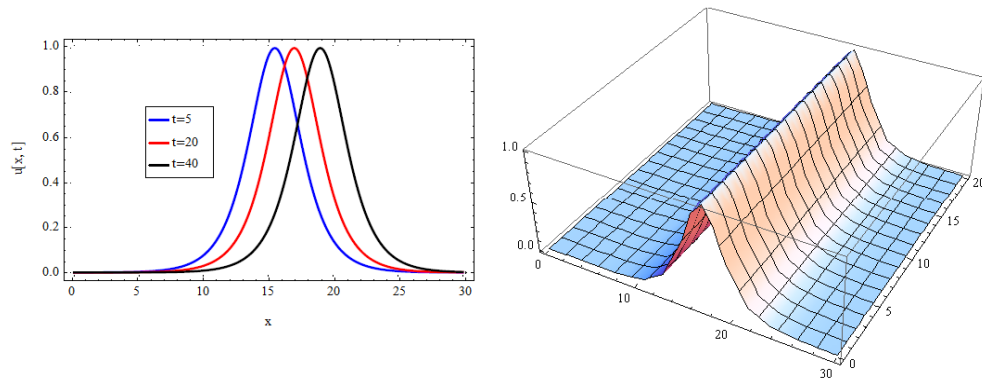


FIGURE 3. Graph of equation (2.11).

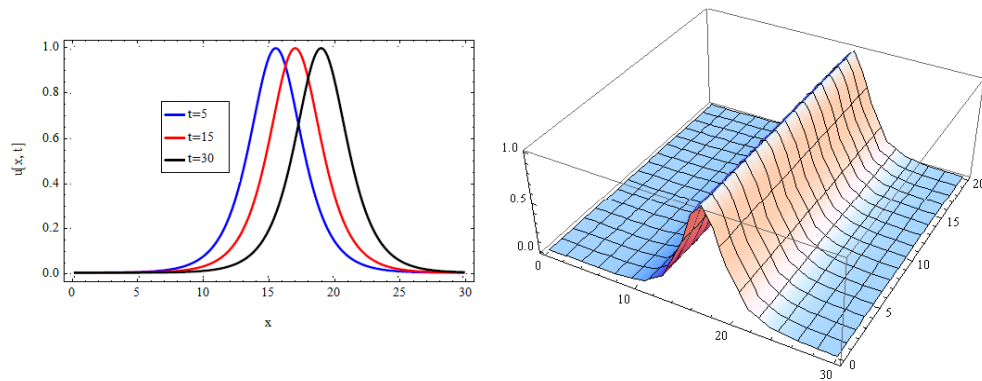


FIGURE 4. Graph of equation (2.18).

in which unknown parameters  $\delta_j(t)$  are procured using boundary and weighted residual conditions. Showing regard to the transformation  $h\eta = x - x_m$  ( $0 \leq \eta \leq 1$ ) the finite interval  $[x_{m+1} - x_m]$  is transform into more easily convenient interval  $[0, 1]$ . So cubic B-spline shape functions (4.1) in terms of  $\eta$  over the domain  $[0, 1]$  can be reformulated as

$$\begin{aligned}
 \phi_{m-1} &= (1 - \eta)^3, \\
 \phi_m &= 1 + 3(1 - \eta) + 3(1 - \eta)^2 - 3(1 - \eta)^3, \\
 \phi_{m+1} &= 1 + 3\eta + 3\eta^2 - 3\eta^3, \\
 \phi_{m+2} &= \eta^3.
 \end{aligned}
 \tag{4.3}$$

Cubic B-splines except  $\phi_{m-1}(x), \phi_m(x), \phi_{m+1}(x), \phi_{m+2}(x)$  and their four principal derivatives vanish the outside of the region  $[0, 1]$ . Hence approximation function (4.2) in terms of element parameters  $\delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2}$  and B-spline element functions



$\phi_{m-1}, \phi_m, \phi_{m+1}, \phi_{m+2}$  are given over the region  $[0, 1]$  by

$$U_N(\eta, t) = \sum_{j=m-1}^{m+2} \delta_j \phi_j. \tag{4.4}$$

Using equalities (4.3) and (4.4), approximation of nodal values  $U_m$  and its first and second derivatives are obtained as follows:

$$\begin{aligned} U_m &= U(x_m) = \delta_{m-1} + 4\delta_m + \delta_{m+1}, \\ U'_m &= U'(x_m) = 3(-\delta_{m-1} + \delta_{m+1}), \\ U''_m &= U''(x_m) = 6(\delta_{m-1} - 2\delta_m + \delta_{m+1}). \end{aligned} \tag{4.5}$$

The Finite element method is a numerical method for the approximate solution of most problems that can be formulated as a system of partial differential equations. Finite element method belongs to the family of weighted residual methods [41]. One of the standard method is Galerkin method [49]. In Galerkin form of weighted residual method, the weight functions are selected to be the trial functions themselves. Hence, in Galerkin method we set

$$W_i = N_i \quad (i = 1, 2, \dots, n) \tag{4.6}$$

The unknown coefficients in the approximate solution are detected by setting the integral over  $D$  of the weighted residual to zero. For one-dimensional problem in the interval  $[a, b]$ , this method will results

$$\int_a^b W_i R(x) dx = \int_a^b N_i R(x) dx = 0 \quad (i = 1, 2, \dots, n), \tag{4.7}$$

Also following points about Galerkin method can be written down:

- Galerkin method generates symmetric positive definite coefficient matrix if the differential operator is self-adjoint.
- Galerkin method needs less computational operation compared to the others method [49].

When  $W(x)$  is taken as the weight function and the Galerkin's method is implemented to Eq.(1.2), weak formulation of Eq.(1.2) is attained as

$$\int_a^b W(U_t + \varepsilon U^p U_x + \mu U_{xxx}) dx = 0. \tag{4.8}$$

Since the Galerkin method are used and in the method the weight function  $W(x)$  is chosen as exactly same as approximate functions and also the approximate functions are chosen as B-splines, the smoothness of the weight function is warranted. If we use the  $h\eta = x - x_m$  transformation, Eq.(4.8) turns into following equation:

$$\int_0^1 W \left( U_t + \varepsilon \left( \frac{U^p}{h} \right) U_\eta + \mu \left( \frac{1}{h^3} \right) U_{\eta\eta\eta} \right) d\eta = 0. \tag{4.9}$$



If partial integration is performed to (4.9), this guides to following equation:

$$\int_0^1 [W(U_t + \varepsilon\kappa U_\eta) - (\xi W_\eta U_{\eta\eta})] d\eta = -\xi W U_{\eta\eta}|_0^1, \quad (4.10)$$

in which  $\kappa = \frac{U^p}{h}$  and  $\xi = \frac{\mu}{h^3}$ . Choosing the weight function as cubic B-spline shape functions indicated by Eq.(4.3) and replacing approximation (4.4) in integral equation (4.10) with some manipulation, we get the element contributions in the form

$$\sum_{j=m-1}^{m+2} \left[ \int_0^1 \phi_i \phi_j d\eta \right] \dot{\delta}_j^e + \sum_{j=m-1}^{m+2} \left[ \varepsilon\kappa \left( \int_0^1 \phi_i \phi_j' d\eta \right) - \left( \xi \int_0^1 \phi_i \phi_j'' d\eta \right) + \left( \xi \phi_i \phi_j''|_0^1 \right) \right] \delta_j^e = 0, \quad (4.11)$$

where  $\delta^e = (\delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2})^T$  and dot represents differentiation to  $t$ , which is given in following matrix form

$$[A^e] \dot{\delta}^e + [(\varepsilon\kappa B^e - \xi(C^e - D^e))] \delta^e = 0. \quad (4.12)$$

Element matrices are

$$A_{ij}^e = \int_0^1 \phi_i \phi_j d\eta = \frac{1}{140} \begin{bmatrix} 20 & 129 & 60 & 1 \\ 129 & 1188 & 933 & 60 \\ 60 & 933 & 1188 & 129 \\ 1 & 60 & 129 & 20 \end{bmatrix},$$

$$B_{ij}^e = \int_0^1 \phi_i \phi_j' d\eta = \frac{1}{20} \begin{bmatrix} -10 & -9 & 18 & 1 \\ -71 & -150 & 183 & 38 \\ -38 & -183 & 150 & 71 \\ -1 & -18 & 9 & 10 \end{bmatrix},$$

$$C_{ij}^e = \int_0^1 \phi_i \phi_j'' d\eta = \frac{1}{2} \begin{bmatrix} -9 & 15 & -3 & -3 \\ -15 & 9 & 27 & -21 \\ 21 & -27 & -9 & 15 \\ 3 & 3 & -15 & 9 \end{bmatrix},$$

$$D_{ij}^e = \phi_i \phi_j''|_0^1 = \begin{bmatrix} -6 & 12 & -6 & 0 \\ -24 & 54 & -36 & 6 \\ -6 & 36 & -54 & 24 \\ 0 & 6 & -12 & 6 \end{bmatrix},$$

with the subscript  $i, j = m-1, m, m+1, m+2$ . A lumped value for  $U$  is got from  $(\frac{U_m + U_{m+1}}{2})^p$  as

$$\lambda = \frac{1}{4h} (\delta_{m-1} + 5\delta_m + 5\delta_{m+1} + \delta_{m+2})^p.$$

By taking in consideration associate supplementations from all elements, matrix equation (4.12) is as the form

$$[A] \dot{\delta} + [(\varepsilon\kappa B - \xi(C - D))] \delta = 0, \quad (4.13)$$



where  $\delta = (\delta_{-1}, \delta_0, \dots, \delta_N, \delta_{N+1})^T$  global element parameters. The  $A, B, C$  and  $\lambda D$  matrices are septa-diagonal and their each line of  $m$  are

$$\begin{aligned}
 A &= \frac{1}{140} (1, 120, 1191, 2416, 1191, 120, 1), \\
 \kappa B &= \frac{1}{20} \begin{pmatrix} -\lambda_1, -18\lambda_1 - 38\lambda_2, 9\lambda_1 - 183\lambda_2 - 71\lambda_3, 10\lambda_1 + 150\lambda_2 - 150\lambda_3 - 10\lambda_4, \\ 71\lambda_2 + 183\lambda_3 - 9\lambda_4, 38\lambda_3 + 18\lambda_4, \lambda_4 \end{pmatrix}, \\
 C &= \frac{1}{2} (3, 24, -57, 0, 57, -24, -3), \\
 D &= (0, 0, 0, 0, 0, 0, 0),
 \end{aligned}$$

where

$$\begin{aligned}
 \kappa_1 &= \frac{1}{4h} (\delta_{m-2} + 5\delta_{m-1} + 5\delta_m + \delta_{m+1})^p, \quad \kappa_2 = \frac{1}{4h} (\delta_{m-1} + 5\delta_m + 5\delta_{m+1} + \delta_{m+2})^p, \\
 \kappa_3 &= \frac{1}{4h} (\delta_m + 5\delta_{m+1} + 5\delta_{m+2} + \delta_{m+3})^p, \quad \kappa_4 = \frac{1}{4h} (\delta_{m+1} + 5\delta_{m+2} + 5\delta_{m+3} + \delta_{m+4})^p.
 \end{aligned}$$

Performing the forward finite difference  $\dot{\delta} = \frac{\delta^{n+1} - \delta^n}{\Delta t}$  and Crank-Nicolson formula  $\delta = \frac{1}{2}(\delta^n + \delta^{n+1})$  to equation (4.13), we acquire following septa-diagonal matrix system

$$[A + \varepsilon \kappa B - \xi(C - D) \frac{\Delta t}{2}] \delta^{n+1} = [A - \varepsilon \kappa B - \xi(C - D) \frac{\Delta t}{2}] \delta^n. \tag{4.14}$$

Practicing the boundary conditions (1.3) to the Eq.(4.14),  $(N + 1) \times (N + 1)$  matrix system is obtained. The result system is effectively solved by using Thomas algorithm. In solution process, two or three inner iterations  $\delta^{n*} = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1})$  are also implemented at each time step to decrease the non-linearity. Consequently, repetition connection between time steps  $n$  and  $n+1$  as an ordinary member of the matrix system (4.14) is obtained as:

$$\begin{aligned}
 \rho_1 \delta_{m-3}^{n+1} + \rho_2 \delta_{m-2}^{n+1} + \rho_3 \delta_{m-1}^{n+1} + \rho_4 \delta_m^{n+1} + \rho_5 \delta_{m+1}^{n+1} + \rho_6 \delta_{m+2}^{n+1} + \rho_7 \delta_{m+3}^{n+1} = \\
 \rho_7 \delta_{m-3}^n + \rho_6 \delta_{m-2}^n + \rho_5 \delta_{m-1}^n + \rho_4 \delta_m^n + \rho_3 \delta_{m+1}^n + \rho_2 \delta_{m+2}^n + \rho_1 \delta_{m+3}^n
 \end{aligned} \tag{4.15}$$

where

$$\begin{aligned}
 \rho_1 &= \frac{1}{140} - \frac{\varepsilon \lambda \Delta t}{240} - \frac{3\xi \Delta t}{4}, \quad \rho_2 = \frac{120}{140} - \frac{56\varepsilon \lambda \Delta t}{240} - \frac{24\xi \Delta t}{4}, \\
 \rho_3 &= \frac{1191}{140} - \frac{245\varepsilon \lambda \Delta t}{240} + \frac{57\xi \Delta t}{4}, \quad \rho_4 = \frac{2416}{140}, \\
 \rho_5 &= \frac{1191}{140} + \frac{245\varepsilon \lambda \Delta t}{240} - \frac{57\xi \Delta t}{4}, \quad \rho_6 = \frac{120}{140} + \frac{56\varepsilon \lambda \Delta t}{240} + \frac{24\xi \Delta t}{4}, \\
 \rho_7 &= \frac{1}{140} + \frac{\varepsilon \lambda \Delta t}{240} + \frac{3\xi \Delta t}{4}.
 \end{aligned}$$

To begin the iteration, the initial vector  $\delta^0$  is calculated by using the initial and boundary conditions. So, using the relations at the knots  $U_N(x_m, 0) = U(x_m, 0)$ ,  $m = 0, 1, 2, \dots, N$  and  $U'_N(x_0, 0) = U'(x_N, 0) = 0$  associated with a variant of the Thomas algorithm, the initial vector  $\delta^0$  is easily found from the following matrix



form

$$\begin{bmatrix} -3 & 0 & 3 & & & \\ 1 & 4 & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & 4 & 1 \\ & & & & -3 & 0 & 3 \end{bmatrix} \begin{bmatrix} \delta_0^0 \\ \delta_1^0 \\ \vdots \\ \delta_N^0 \\ \delta_{N+1}^0 \end{bmatrix} = \begin{bmatrix} U'(x_0, 0) \\ U(x_0, 0) \\ \vdots \\ U(x_N, 0) \\ U'(x_N, 0) \end{bmatrix}.$$

## 5. STABILITY ANALYSIS

In order to examine the stability analysis of the suggested algorithm, it is convenient to use Von-Neumann theory. Supposing  $U^p$  in the nonlinear term  $U^p U_x$  of GKdV equation (1.2) is locally constant. If we put the Fourier mode  $\delta_j^n = \zeta^n e^{i\sigma m h}$ , ( $i = \sqrt{-1}$ ) into the form of (4.15) with some arrangements, the growth factor is derived as

$$\zeta = \frac{X - iY}{X + iY}, \quad (5.1)$$

where  $\sigma$  is mode number and  $h$  is element greatness,  $\theta = \sigma h$ . In Eq.(5.1)

$$\begin{aligned} X &= 14496 + 8292 \cos(\theta) + 1440 \cos(2\theta) + 12 \cos(3\theta), \\ Y &= [-6000 + (1715\varepsilon\kappa - 23940\xi)\Delta t] \sin(\theta) + [(392\varepsilon\kappa + 10080\xi)\Delta t] \sin(2\theta) + \\ & \quad [(7\varepsilon\kappa + 1260\xi)\Delta t] \sin(3\theta). \end{aligned} \quad (5.2)$$

Since  $|\zeta|$  is 1, the von Neumann necessary criterion is provided so our linearized scheme is neutrally stable.

## 6. NUMERICAL RESULTS AND DISCUSSION

The numerical method outlined in the former section is assayed for the propagation of single solitary waves for the GKdV equation by using the homogenous boundary conditions, are investigated. For the test problem, we have calculated the numerical solution of the GKdV equation for  $p = 1, 2$  and 3. The  $L_2$

$$L_2 = \|U^{exact} - U_N\|_2 \simeq \sqrt{h \sum_{j=0}^N |U_j^{exact} - (U_N)_j|^2},$$

and  $L_\infty$

$$L_\infty = \|U^{exact} - U_N\|_\infty \simeq \max_j |U_j^{exact} - (U_N)_j|,$$

error norms are considered to measure difference between analytical and numerical solutions at some specified times and to compare our results with other results in the literature whenever available. The analytical solution of GKdV equation is given in [3, 31] as



$$U(x, t) = A \operatorname{sech}^2[k(x - x_0 - ct)]^{\frac{1}{p}},$$

where  $A = [\frac{c(p+1)(p+2)}{2\varepsilon}]$  and  $k = \frac{p}{2}\sqrt{\frac{c}{\mu}}$ . GKdV equation possesses many invariant polynomials can be derived easily as shown in the following cases

$$\begin{aligned} I_1 &= \int_a^b U(x, t) dx, & I_2 &= \int_a^b [U^2(x, t)] dx, \\ I_3 &= \int_a^b [U^{p+2}(x, t) - \frac{\mu(p+1)(p+2)}{2\varepsilon} (U_x(x, t))^2] dx \end{aligned} \tag{6.1}$$

In the simulation of solitary wave motion, the invariants  $I_1$ ,  $I_2$  and  $I_3$  are also observed to check the accuracy of the numerical algorithm.

**6.1. Propagation of a single solitary wave.** In this section, different numerical examples will be given to illustrate the efficiency and accuracy of the method. For the numerical simulations of the movement of single solitary wave, three sets of parameters have been taken and discussed. For the GKdV equation, parameters used by earlier authors to obtain their results are taken as guiding principle for our calculations.

**6.2. Case 1.** For the first case, the motion of the single solitary wave is modelled with two sets of parameters,  $p = 1, \varepsilon = 1, \mu = 4.84 \times 10^{-4}, c = 0.3, h = 0.01, \Delta t = 0.005, x \in [0, 2]$  and  $\varepsilon = 3, \mu = 1, c = 0.3, h = 0.1, \Delta t = 0.01, x \in [0, 80]$  to make a comparison with the previous papers [3, 12, 20, 21, 24, 53, 54, 55, 63]. These parameters represent the motion of a single solitary wave with amplitude 0.9 and 0.3 respectively and the program is run up to time  $t = 1$  over the solution domains. We calculate the values of the error norms  $L_2, L_\infty$  and invariants  $I_1, I_2$ , and  $I_3$  for different time levels and compare them with earlier papers in Table (1). This table indicates that the error norms obtained by our method are found much better than most of the others and the calculated values of invariants are in good conformity with the earlier. We have found the change of the values of the invariants  $0, 0, 2.8 \times 10^{-5}$  for  $\mu = 4.84 \times 10^{-4}$ ;  $2 \times 10^{-6}, 0, 0$  for  $\mu = 1$  and the error norms  $L_2$  and  $L_\infty$  remain less than  $0.922721 \times 10^{-3}$  and  $2.788981 \times 10^{-3}$  for  $\mu = 4.84 \times 10^{-4}$  and  $0.018 \times 10^{-3}, 0.017 \times 10^{-3}$  for  $\mu = 1$ . Our invariants are almost stable as time increases and the agreement between other solutions is perfect. Hence our method is acceptedly conservative. Solitary wave profiles are demonstrated at different time levels in Fig.(5) in which the soliton moves to the right at a nearly unchanged speed and amplitude as time increases, as expected. The distribution of errors at time  $t = 1$  are designed in Fig.(6). The error deviations for different value of  $\mu$  varies from  $-3 \times 10^{-3}$  to  $4 \times 10^{-3}$  and  $-2 \times 10^{-5}$  to  $5 \times 10^{-6}$ , respectively.

**6.3. Case 2.** For the second case, we have selected the parameters  $p = 2, \varepsilon = 3, \mu = 1, h = 0.1, \Delta t = 0.01, c = 0.845$  and  $c = 0.3, h = 0.1, \Delta t = 0.01, x \in [0, 80]$  to coincide with that of earlier papers [1, 2, 8, 32, 33, 44]. These parameters stand for the motion of a single solitary wave with amplitude 1.3416 and 0.7746 and the computations are done until time  $t = 20$  and  $t = 1$ . Values of the three invariants



TABLE 1. Comparisons of results for invariants and error norms with  $p = 1$ ,  $\varepsilon = 1$ ,  $\mu = 4.84 \times 10^{-4}$ ,  $c = 0.3$ ,  $h = 0.01$ ,  $\Delta t = 0.005$ ,  $x \in [0, 2]$  and  $\varepsilon = 3$ ,  $\mu = 1$ ,  $c = 0.3$ ,  $h = 0.1$ ,  $t = 0.01$ ,

Method	Time	$I_1$	$I_2$	$I_3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
$\mu = 4.84 \times 10^{-4}$ Present Method	0.00	0.144598	0.086759	0.046850	0	0
	0.25	0.144598	0.086759	0.046758	0.403564	0.906736
	0.50	0.144598	0.086759	0.046283	0.545518	1.539350
	0.75	0.144598	0.086759	0.046693	0.720009	2.163753
	1.00	0.144598	0.086759	0.046878	0.922721	2.788981
	[63]				28.66	
[54] Septic Coll.	1.00	0.14460	0.086759	0.046877	22.1	
[55]	1.00	0.144592	0.086759	0.016870	22.2	
[53] P-G	1.00				0.75	
[53] Modified P-G	1.00				4.33	
[24]	1.00				18.72	
[20]	1.00				29.45	
[21]	1.00				63.72	
[12] MQ	1.00	0.144606	0.086759	0.046850	0.062	0.133
[12] IMQ	1.00	0.144623	0.086765	0.046847	2.751	5.018
[12] IQ	1.00	0.144598	0.086759	0.046849	1.013	2.090
[12] TPS	1.00	0.144261	0.086762	0.046842	2.606	6.345
[12] G	1.00	0.144601	0.086760	0.046850	0.046	0.136
[3]	1.00	0.144599	0.086759	0.046850	0.079	0.238
$\mu = 1$ Present Method	0.00	2.190842	0.438176	0.078871	0	0
	0.25	2.190844	0.438176	0.078871	0.013	0.020
	0.50	2.190844	0.438176	0.078871	0.015	0.019
	0.75	2.190844	0.438176	0.078871	0.016	0.018
	1.00	2.190844	0.438176	0.078871	0.018	0.017

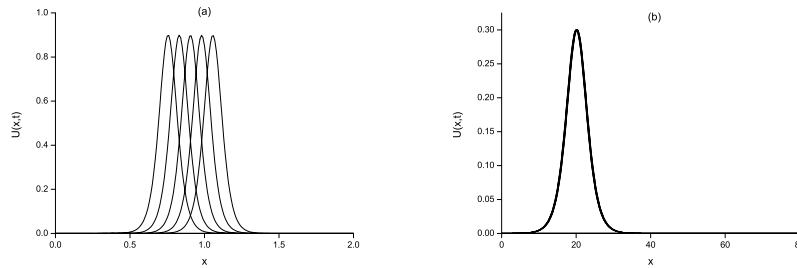


FIGURE 5. Motion of single solitary wave for a)  $p = 1$ ,  $\varepsilon = 1$ ,  $\mu = 4.84 \times 10^{-4}$ ,  $c = 0.3$ ,  $h = 0.01$ ,  $\Delta t = 0.005$  and b)  $\varepsilon = 3$ ,  $\mu = 1$ ,  $c = 0.3$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ .

as well as the error norms have been computed and compared in Table (2). It is noticeably seen from Table (2) that the error norms obtained by our method are in good agreement with the others and the error norms  $L_2$  and  $L_\infty$  remain less than  $1.983089 \times 10^{-3}$ ,  $1.309575 \times 10^{-3}$  for  $c = 0.845$  and  $0.105 \times 10^{-3}$ ,  $0.051 \times 10^{-3}$  for  $c = 0.3$ ; the invariants  $I_1$ ,  $I_2$ , and  $I_3$  change from their initial values by less than 0; 0 and  $2.62 \times 10^{-3}$  for  $c = 0.845$  and  $3 \times 10^{-6}$ ,  $0, 4 \times 10^{-5}$  for  $c = 0.3$ , respectively



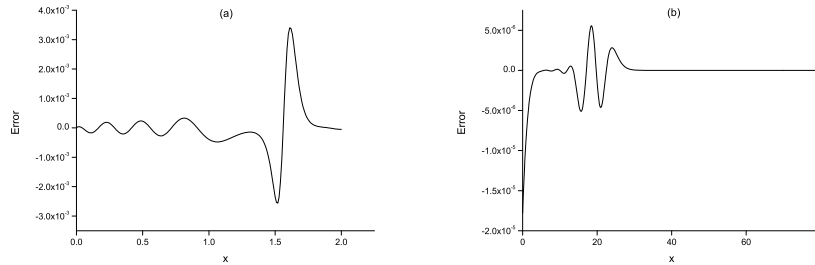


FIGURE 6. Error distributions at  $t = 1$  for the parameters a)  $p = 1$ ,  $\varepsilon = 1$ ,  $\mu = 4.84 \times 10^{-4}$ ,  $c = 0.3$ ,  $h = 0.01$ ,  $\Delta t = 0.005$  and b)  $\varepsilon = 3$ ,  $\mu = 1$ ,  $c = 0.3$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ .

throughout the simulation. Also, our invariants are almost constant as time increases and the change of the invariants agree with the earlier. So we can say our method is sensibly conservative. The motion of solitary wave using our scheme are plotted at times  $t = 0, 5, 10, 15, 20$  and  $t = 0, 0.1, 0.2, \dots, 1$  in Fig.(7). It is obvious from the figure that the suggested method performs the motion of propagation of a solitary wave admissibly, which moved to the right with the preserved amplitude and shape. To demonstrate the errors between the exact and numerical results over the solution domain, error distributions at time  $t = 20$  and  $t = 1$  is depicted graphically in Figure(8). The maximum errors are between  $-1 \times 10^{-3}$  to  $1.5 \times 10^{-3}$  and  $-6 \times 10^{-5}$  to  $6 \times 10^{-5}$ , respectively and occur around the central position of the solitary wave.

TABLE 2. Comparisons of results for invariants and error norms with  $p = 2$ ,  $\varepsilon = 3$ ,  $\mu = 1$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ ,  $c = 0.845$  and  $c = 0.3$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ .

Method	Time	$I_1$	$I_2$	$I_3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	
$c = 0.845$ Present Method	0	4.442865	3.676941	2.071335	0	0	
	5	4.442865	3.676941	2.073758	0.917706	0.562852	
	10	4.442865	3.676941	2.073900	1.265494	0.850150	
	15	4.442865	3.676941	2.073930	1.638275	1.096426	
	20	4.442865	3.676941	2.073948	1.983089	1.309575	
	20	4.443171	3.679192	2.077161	-	8.642137	
[8]	20	4.442866	3.676941	2.073841	3.656694	2.294197	
[1]	20	4.442866	3.676941	2.073846	3.641638	2.285638	
[2]	1	4.442863	3.676933	2.071312	830.4	480.5	
[32]	1	4.442865	3.676941	2.071327	0.0184	0.0117	
[33]	1	4.442865	3.676941	2.071327	0.0184	0.0117	
$c = 0.3$ Present Method	0.00	4.442815	2.190881	0.438173	0	0	
	0.25	4.442818	2.190881	0.438179	0.046	0.030	
	0.50	4.442818	2.190881	0.438191	0.073	0.042	
	0.75	4.442818	2.190881	0.438202	0.091	0.048	
	1.00	4.442818	2.190881	0.438213	0.105	0.051	
	[44] First Scheme	1.00	4.44192	2.18994	0.437763	-	0.310
	[44] Second Scheme	1.00	4.44198	2.18974	0.437642	-	0.325
	[32]	1.00	4.442765	2.190882	0.438173	140.1	635.5
[33]	1.00	4.44285	2.1908	0.438146	0.107	0.200	





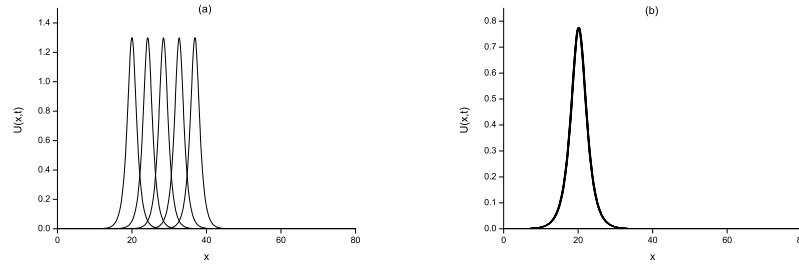


FIGURE 7. Motion of single solitary wave for a)  $p = 2$ ,  $\varepsilon = 3$ ,  $\mu = 1$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ ,  $c = 0.845$  and b)  $c = 0.3$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ .

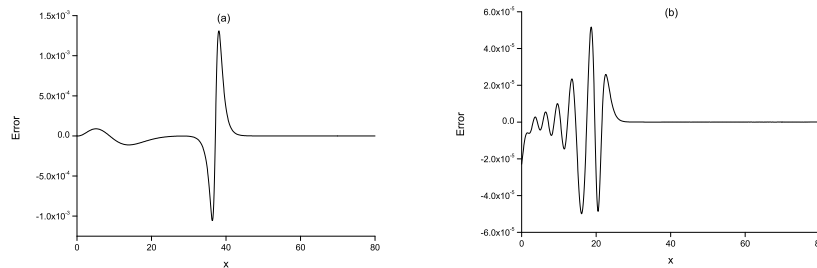


FIGURE 8. Error distributions for the parameters a)  $p = 2$ ,  $\varepsilon = 3$ ,  $\mu = 1$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ ,  $c = 0.845$ ,  $t = 20$  and  $c = 0.3$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ ,  $t = 1$ .

**6.4. Case 3.** For the last case, we have taken the parameters  $p = 3$ ,  $\varepsilon = 3$ ,  $\mu = 1$ ,  $h = 0.01$ ,  $\Delta t = 0.005$ ,  $c = 0.845$  and  $c = 0.3$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ ,  $x \in [0, 80]$ . These values yield the amplitude 1.4122 and 1.0000, respectively and the run of the algorithm is continued up to time  $t = 20$  and  $t = 1$  over the solution regions. The error norms  $L_2$ ,  $L_\infty$  and conservation quantities  $I_1$ ,  $I_2$ , and  $I_3$  are computed and given in Table (3). It can be noted from Table (3); the error norms  $L_2$  and  $L_\infty$  remain less than  $9.150918 \times 10^{-3}$ ,  $6.747899 \times 10^{-3}$  for  $c = 0.845$  and  $0.234 \times 10^{-3}$ ,  $0.127 \times 10^{-3}$  for  $c = 0.3$ ; the invariants  $I_1$ ,  $I_2$ , and  $I_3$  change from their initial values by less than 0, 0 and  $2.62 \times 10^{-4}$  for  $c = 0.845$  and  $3 \times 10^{-6}$ ,  $0.4 \times 10^{-5}$  for  $c = 0.3$  respectively, throughout the simulation. Also, we have found out error norms  $L_2$  and  $L_\infty$  are obtained sufficiently small during the computer run and our invariants are almost constant as time increases. Therefore we can say our method is marginally conservative. For visual representation, behaviours of solutions at times



$t = 0, 5, 10, 15, 20$  and  $t = 0, 0.1, \dots, 1$  are depicted in Figure(9). We observed from the Figure(9) that single soliton travels to the right at a constant speed and keeps its amplitude and form with increasing time as not surprisingly. Error distributions at time  $t = 20$  and  $t = 1$  are shown graphically in Figure(10). As it is seen, the maximum errors occur around the central position of the solitary wave and between  $-3 \times 10^{-2}$  to  $3 \times 10^{-2}$  and  $-1.5 \times 10^{-4}$  to  $1.5 \times 10^{-4}$ .

TABLE 3. Values of the invariants and error norms for  $p = 3, \varepsilon = 3, \mu = 1, h = 0.01, \Delta t = 0.005, c = 0.845$  and  $c = 0.3, h = 0.1, \Delta t = 0.01$ .

Method	Time	$I_1$	$I_2$	$I_3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
$c = 0.845$ Present Method	0	4.308401	3.742115	1.505761	0	0
	5	4.308401	3.742115	1.515888	1.901952	1.326051
	10	4.308401	3.742115	1.517054	4.240384	2.930073
	15	4.308401	3.742115	1.517358	6.117633	4.778102
	20	4.308401	3.742115	1.517459	9.150918	6.747899
$c = 0.3$ Present Method	0.00	5.119973	3.148917	0.449836	0	0
	0.25	5.119976	3.148917	0.449836	0.131	0.091
	0.50	5.119976	3.148917	0.449835	0.183	0.109
	0.75	5.119976	3.148917	0.450107	0.213	0.116
	1.00	5.119976	3.148917	0.450165	0.234	0.127

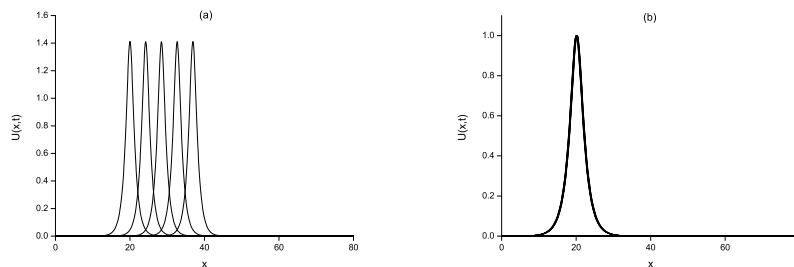


FIGURE 9. Motion of single solitary wave for a)  $p = 3, \varepsilon = 3, \mu = 1, h = 0.01, \Delta t = 0.005, c = 0.845$  and b)  $c = 0.3, h = 0.1, \Delta t = 0.01$ .

### 7. CONCLUSION

In this paper, GKdV equation is comprehensively investigated by using two credible methods. The first method is the solitary wave ansatz method that holds notable properties that make it useful for the determination of single soliton solutions for an extensive class of nonlinear evolution equations with constant and varying coefficients in a direct method while the second method is a lumped Galerkin finite element method also gives the soliton solution of the equation. The best part of the study is successful implementation of both the schemes for finding both exact and numerical



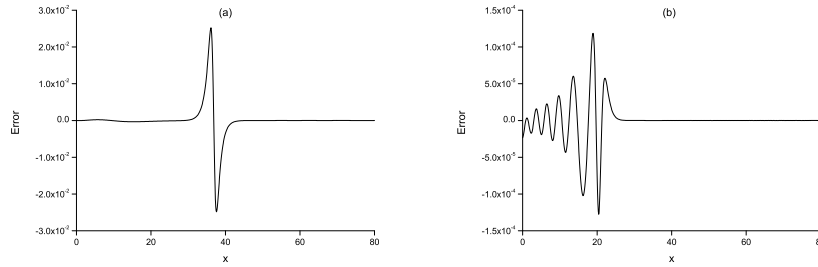


FIGURE 10. Error distributions for the parameters a)  $p = 3$ ,  $\varepsilon = 3$ ,  $\mu = 1$ ,  $h = 0.01$ ,  $\Delta t = 0.005$ ,  $c = 0.845$ ,  $t = 20$  and b)  $c = 0.3$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ ,  $t = 1$ .

results. For the two methods we portray some graphical illustrations of the obtained solutions of the GKdV equation. To prove the performance of the numerical algorithm, the error norms  $L_2$ ,  $L_\infty$  and the invariants  $I_1$ ,  $I_2$ , and  $I_3$  have been calculated. The newly suggested numerical scheme produces highly accurate results and the conserved quantities are almost constant during the simulation for all cases. We can also see that our numerical scheme for the equation is more accurate than the some other earlier schemes found in the literature. Stability analysis have been done and the method is shown to be unconditionally stable. As a result, we can say that our exact and numerical techniques are more practical, accurate and powerful mathematical tool for solving nonlinear partial differential equations having wide applications in physical problem represented by GKdV equation.



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