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A numerical technique based on collocation method for solving modified Kawahara equation

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Abstract

In this article, a numerical solution of the modified Kawahara equation is presented by septic B-spline collocation method. Applying the von-Neumann stability analysis, the present method is shown to be unconditionally stable. L_2 and L_∞ error norms and conserved quantities are given at selected times. The accuracy of the proposed method is checked by test problems including motion of the single solitary wave, interaction of solitary waves and evolution of solitons.

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Keywords: Modified Kawahara equation; Finite element method; Collocation; Solitary waves; B-spline.

1. Introduction

The dynamics of shallow water waves is a seriously growing research area in the area of fluid dynamics. There are several models that are available to focus on this area of research. A few well known models that are frequently visible in journals and text books are Korteweg–de Vries (KdV) equation [1–3], modified KdV equation [4–8], Peregrine equation [9], Benjamin–Bona–Mahony equation [10–12], Boussinesq equation [13,14] and many more [15,16]. For coupled system, where two-layered shallow water waves are studied, several models are proposed there too. These are Bona–Chen equation [17], Gear–Grimshaw model [18,19], Zahreamoghaddam model [20] and others. This paper will focus on single layered shallow water fluids that are observed along sea shores and beaches. The model that will be focused in this paper is known as modified Kawahara equation (mKE).

Modified Kawahara equation has been studied in the past during several occasions [21–28]. This equation models water waves in long wave regime for moderate values of surface tension [22]. It was analyzed that KdV equation fails in this context since the cubic term in dispersion relation vanishes and thus fifth order dispersion terms stands relevant. The detail of this analysis is reported during 2006 [22].

This paper discusses the numerical analysis of modified Kawahara equation using septic B-splines. The motion of single solitary waves is analyzed in details using this form of numerical simulations. Interaction of two and three solitary waves, stability analysis are all detailed in this paper with initial wave form being Gaussian type.

2. The governing equation and septic B-splines

In this study, we will consider the modified Kawahara equation

$$U_t + \alpha U^2 U_x + \beta U_{xxx} - \gamma U_{xxxxx} = 0, \tag{1}$$

with the physical boundary conditions $U \to 0$ as $x \to \pm\infty$, where α, β and γ are positive parameters and the subscripts x and t denote the differentiation. To implement the numerical method, solution domain is restricted over an interval $a \le x \le b$. Boundary conditions will be selected from the

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Table 1
Septic B-spline function and its derivatives at nodes x_m.

x	x_{m-4}	x_{m-3}	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}	x_{m+3}	x_{m+4}
$\phi_m(x)$	0	1	120	1191	2416	1191	120	1	0
$h\phi'_m(x)$	0	−7	−392	−1715	0	1715	392	7	0
$h^2\phi''_m(x)$	0	42	1008	630	−3360	630	1008	42	0
$h^3\phi'''_m(x)$	0	−210	−1680	3990	0	−3990	1680	210	0
$h^4\phi^{iv}_m(x)$	0	840	0	−7560	13440	−7560	0	840	0
$h^5\phi^{v}_m(x)$	0	−2520	10080	−12600	0	12600	−100800	2520	0
$h^6\phi^{vi}_m(x)$	0	5040	−30240	75600	−100800	75600	−30240	5040	0

following homogeneous boundary conditions:

$$\begin{array}{lll} U(a,t)=0, & U(b,t)=0, & \\ U_x(a,t)=0 & U_x(b,t)=0, & \\ U_{xxx}(a,t)=0, & U_{xxx}(b,t)=0, & t>0 \end{array} \tag{2}$$

and the initial condition

$$U(x,0)=f(x),\ a \quad \le x \le b. \tag{3}$$

The septic B-splines $\phi_m(x)$, ($m= -3(1)N+3$), at the knots x_m are defined over the interval [a, b] by Ref. [29].

$$\phi_m(x) = \frac{1}{h^7}\begin{cases} (x-x_{m-4})^7, & [x_{m-4},x_{m-3}] \\ (x-x_{m-4})^7-8(x-x_{m-3})^7, & [x_{m-3},x_{m-2}] \\ (x-x_{m-4})^7-8(x-x_{m-3})^7 \\ \quad +28(x-x_{m-2})^7, & [x_{m-2},x_{m-1}] \\ (x-x_{m-4})^7-8(x-x_{m-3})^7 \\ \quad +28(x-x_{m-2})^7-56(x-x_{m-1})^7, & [x_{m-1},x_m] \\ (x_{m+4}-x)^7-8(x_{m+3}-x)^7 \\ \quad +28(x_{m+2}-x)^7-56(x_{m+1}-x)^7, & [x_m,x_{m+1}] \\ (x_{m+4}-x)^7-8(x_{m+3}-x)^7 \\ \quad +28(x_{m+2}-x)^7, & [x_{m+1},x_{m+2}] \\ (x_{m+4}-x)^7-8(x_{m+3}-x)^7, & [x_{m+2},x_{m+3}] \\ (x_{m+4}-x)^7, & [x_{m+3},x_{m+4}] \\ 0. & elsewhere. \end{cases} \tag{4}$$

The set of functions $\{\phi_{-3}(x),\phi_{-2}(x),\phi_{-1}(x),\ldots,\phi_{N+1}(x),\phi_{N+2}(x),\phi_{N+3}(x)\}$ forms a basis for functions defined over [a, b]. The approximate solution $U_N(x,\ t)$ to the exact solution $U(x,\ t)$ is given by

$$U_N(x,t)=\sum_{i=-3}^{N+3}\phi_i(x)\delta_i(t), \tag{5}$$

where $\delta_i(t)$ are time dependent parameters to be determined from the boundary and weighted residual conditions. Each septic B-spline covers eight elements so that each element $[x_m,x_{m+1}]$ is covered by eight splines. The values of $\phi_m(x)$ and its derivative may be tabulated as in Table 1.

Using trial function (5) and septic splines (4), the values of U, U', U'', U''', U^{iv}, U^{v}, U^{vi} at the knots are determined in terms of the element parameters δ_m by

$$\begin{aligned} U_m &= U(x_m)=\delta_{m-3}-6\delta_{m-2}+15\delta_{m-1}-20\delta_m \\ &\quad +15\delta_{m+1}-6\delta_{m+2}+\delta_{m+3}, \\ U'_m &= U'(x_m)=\frac{7}{h}(-\delta_{m-3}-56\delta_{m-2}-245\delta_{m-1} \\ &\quad +245\delta_{m+1}+56\delta_{m+2}+\delta_{m+3}), \\ U''_m &= U''(x_m)=\frac{42}{h^2}(\delta_{m-3}+24\delta_{m-2}+15\delta_{m-1} \\ &\quad -80\delta_m+15\delta_{m+1}+24\delta_{m+2}+\delta_{m+3}), \\ U'''_m &= U'''(x_m)=\frac{210}{h^3}(-\delta_{m-3}-8\delta_{m-2}+19\delta_{m-1} \\ &\quad -19\delta_{m+1}+8\delta_{m+2}+\delta_{m+3}), \\ U^{iv}_m &= U^{iv}(x_m)=\frac{840}{h^4}(\delta_{m-3}-9\delta_{m-1}+16\delta_m-9\delta_{m+1} \\ &\quad +\delta_{m+3}), \\ U^{v}_m &= U^{v}(x_m)=\frac{2520}{h^5}(-\delta_{m-3}+4\delta_{m-2}-5\delta_{m-1}+5\delta_{m+1} \\ &\quad -4\delta_{m+2}+\delta_{m+3}), \\ U^{vi}_m &= U^{vi}(x_m)=\frac{5040}{h^6}(\delta_{m-3}-6\delta_{m-2}+15\delta_{m-1}-20\delta_m \\ &\quad +15\delta_{m+1}-6\delta_{m+2}+\delta_{m+3}). \end{aligned} \tag{6}$$

where the symbols $'$, $''$, $'''$, ^{iv}, ^{v} and ^{vi} denotes differentiation with respect to x, respectively. The splines $\phi_m(x)$ and its two principle derivatives vanish outside the interval $[x_{m-4},x_{m+4}]$.

3. Collocation finite element method

Now, we identify the collocation points with the knots and use Eq. (6) to evaluate U_m, its necessary space derivatives and substitute into Eq. (1) to obtain the set of the coupled ordinary differential equations. For the linearization technique we get the following equation:

$$\begin{aligned} &\dot\delta_{m-3}+120\dot\delta_{m-2}+1191\dot\delta_{m-1}+2416\dot\delta_m+1191\dot\delta_{m+1} \\ &\quad +120\dot\delta_{m+2}+\dot\delta_{m+3} \\ &\quad +\frac{7\alpha Z_m}{h}(-\delta_{m-3}-56\delta_{m-2}-245\delta_{m-1}+245\delta_{m+1} \\ &\quad +56\delta_{m+2}+\delta_{m+3}) \\ &\quad +\frac{210\beta}{h^3}(-\delta_{m-3}-8\delta_{m-2}+19\delta_{m-1}-19\delta_{m+1}+8\delta_{m+2} \\ &\quad +\delta_{m+3}) \\ &\quad -\frac{2520\gamma}{h^5}(-\delta_{m-3}+4\delta_{m-2}-5\delta_{m-1}+5\delta_{m+1}-4\delta_{m+2} \\ &\quad +\delta_{m+3})=0. \end{aligned} \tag{7}$$

where

$$\begin{aligned} Z_m &= (\delta_{m-3}+120\delta_{m-2}+1191\delta_{m-1}+2416\delta_m+1191\delta_{m+1} \\ &\quad +120\delta_{m+2}+\delta_{m+3})^2, \end{aligned} \tag{8}$$

and $\cdot$ denotes derivative with respect to time. If time parameters δ_i's and its time derivatives $\dot\delta_i$'s in Eq. (7) are discretized by the Crank–Nicolson formula and usual finite difference approximation, respectively,

$$\delta_i = \frac{\delta_i^{n+1} + \delta_i^n}{2}, \qquad \dot{\delta}_i = \frac{\delta_i^{n+1} - \delta_i^n}{\Delta t} \tag{9}$$

we obtain a recurrence relationship between two time levels n and $n+1$ relating two unknown parameters δ_i^{n+1}, δ_i^n for $i = m-3, m-2, \ldots, m+2, m+3$

$$\begin{aligned} &\gamma_1 \delta_{m-3}^{n+1} + \gamma_2 \delta_{m-2}^{n+1} + \gamma_3 \delta_{m-1}^{n+1} + \gamma_4 \delta_m^{n+1} + \gamma_5 \delta_{m+1}^{n+1} + \gamma_6 \delta_{m+2}^{n+1} \\ &\quad + \gamma_7 \delta_{m+3}^{n+1} \\ &= \gamma_7 \delta_{m-3}^n + \gamma_6 \delta_{m-2}^n + \gamma_5 \delta_{m-1}^n + \gamma_4 \delta_m^n + \gamma_3 \delta_{m+1}^n + \gamma_2 \delta_{m+2}^n \\ &\quad + \gamma_1 \delta_{m+3}^n, \end{aligned} \tag{10}$$

where

$$\begin{aligned} \gamma_1 &= [1 - EZ_m - M + K], \\ \gamma_2 &= [120 - 56EZ_m - 8M - 4K], \\ \gamma_3 &= [1191 - 245EZ_m + 19M + 5K], \\ \gamma_4 &= [2416], \\ \gamma_5 &= [1191 + 245EZ_m - 19M - 5K], \\ \gamma_6 &= [120 + 56EZ_m + 8M + 4K], \\ \gamma_7 &= [1 + EZ_m + M - K], \end{aligned}$$

$$m = 0, 1, \ldots, N, \quad E = \frac{7}{2h}\alpha \Delta t, \quad M = \frac{105}{h^3}\beta \Delta t, \quad \times K = \frac{840}{h^4}\gamma \Delta t. \tag{11}$$

$$W = \begin{bmatrix} 1536 & 2712 & 768 & 24 & & & & & & \\ \frac{82731}{81} & \frac{210568.5}{81} & \frac{104796}{81} & \frac{10063.5}{81} & 1 & & & & & \\ \frac{9600}{81} & \frac{96597}{81} & \frac{195768}{81} & \frac{96474}{81} & 120 & 1 & & & & \\ 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & & & \\ & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & & \\ & & & & & \ddots & & & & \\ & & & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 \\ & & & & 1 & 120 & \frac{96474}{81} & \frac{195768}{81} & \frac{96597}{81} & \frac{9600}{81} \\ & & & & & 1 & \frac{10063.5}{81} & \frac{104796}{81} & \frac{210568.5}{81} & \frac{82731}{81} \\ & & & & & & 24 & 768 & 2712 & 1536 \end{bmatrix}$$

$$d^0 = (\delta_0, \delta_1, \delta_2, \ldots, \delta_{N-2}, \delta_{N-1}, \delta_N)^T \quad \text{and} \quad C = [U(x_0, 0), U(x_1, 0), \ldots, U(x_{N-1}, 0), U(x_N, 0)]^T.$$

For the linearization technique, the term U^2 in non-linear term $U^2 U_x$ is taken as

$$\begin{aligned} Z_m = U_m^2 = (&\delta_{m-3} + 120\delta_{m-2} + 1191\delta_{m-1} + 2416\delta_m \\ &+ 1191\delta_{m+1} + 120\delta_{m+2} + \delta_{m+3})^2. \end{aligned} \tag{12}$$

The system (10) consists of $(N+1)$ linear equations including $(N+7)$ unknown parameters $(\delta_{-3}, \delta_{-2}, \delta_{-1}, \ldots, \delta_{N+1}, \delta_{N+2}, \delta_{N+3})^T$. To obtain a unique solution to this system, we need six additional constraints. These are obtained from the boundary conditions and can be used to eliminate δ_{-3}, δ_{-2}, δ_{-1} and δ_{N+1}, δ_{N+2}, δ_{N+3} from the system (10) which then becomes a matrix equation for the $N+1$ unknowns $d = (\delta_0, \delta_1, \ldots, \delta_N)^T$ of the form

$$A\mathbf{d}^{\mathbf{n+1}} = B\mathbf{d}^{\mathbf{n}}. \tag{13}$$

The matrices A and B are septa-diagonal $(N+1) \times (N+1)$ matrices and so they are easily solved. Two or three inner iterations are applied to the term $\delta^{n*} = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1})$ at each time step to cope with the non-linearity caused by Z_m. Before the commencement of the solution process, initial parameters d^0 must be determined by using the initial condition and following derivatives at the boundaries;

$$\begin{aligned} &U_N(x, 0) = U(x_m, 0); && m = 0, 1, 2, \ldots, N \\ &(U_N)_x(a, 0) = 0, && (U_N)_x(b, 0) = 0, \\ &(U_N)_{xx}(a, 0) = 0, && (U_N)_{xx}(b, 0) = 0, \\ &(U_N)_{xxx}(a, 0) = 0, && (U_N)_{xxx}(b, 0) = 0. \end{aligned} \tag{14}$$

So we have the following matrix form for the initial vector d^0;

$$Wd^0 = C, \tag{15}$$

where

4. Stability analysis

The stability analysis is based on the von Neumann theory. The growth factor ξ of the error in a typical mode of amplitude

$$\delta_m^n = \xi^n e^{imkh}, \tag{16}$$

where k is the mode number and h the element size, is determined from a linearization of the numerical scheme. In order to apply the stability analysis, the Kawahara equation can be linearized by assuming that the quantity U^2 in the non-linear term $U^2 U_x$ is locally constant. Substituting the Fourier mode (16) into (10) gives the growth factor ξ of the form

$$\xi = \frac{a - ib}{a + ib}, \tag{17}$$

where

Table 2
Invariants and error norms for single solitary wave with $\alpha=\beta=\gamma=1$, $h=0.1$ and $\Delta t=0.05$.

t	I_1	I_2	$L_2-Error$	$L_\infty-Error$
0	8.4852829335	5.3665642087	.0000000000	.0000000000
20	8.4852829848	5.3665642087	2.747538e-6	1.260529e-6
40	8.4852830757	5.3665642086	4.045791e-6	1.620782e-6
60	8.4852821848	5.3665642087	5.629183e-6	2.617156e-6
80	8.4852800315	5.3665642087	6.811939e-6	2.680477e-6
100	8.4852730550	5.3665642087	8.275389e-6	2.909858e-6

$$\begin{aligned}a&=(\gamma_3+\gamma_5)\cos[hk]+(\gamma_2+\gamma_6)\cos[2hk]+(\gamma_1+\gamma_7)\\&\quad\times\cos[3hk]+\gamma_4,\\b&=(\gamma_7-\gamma_1)\sin[hk]+(\gamma_6-\gamma_2)\sin[2hk]+(\gamma_5-\gamma_3)\\&\quad\times\cos[3hk].\end{aligned}\tag{18}$$

The modulus of $|\xi|$ is 1, therefore the linearized scheme is unconditionally stable.

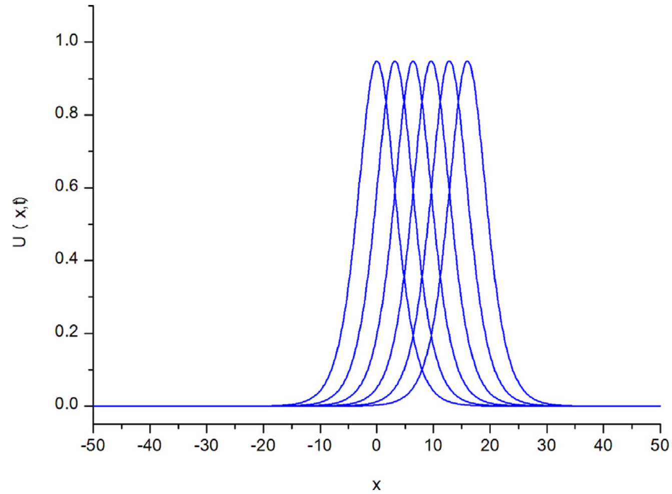

Fig. 1. Single solitary wave profile for $\alpha=\beta=\gamma=1$, $h=0.1$, $\Delta t=0.05$ and $0\le t\le 100$.

5. Numerical examples and results

Numerical results of the mKdV equation are obtained for four problems: the motion of single solitary wave, interaction of two and three solitary waves and evolution of solitons. We use the error norm L_2

$$L_2=\left\|U^{exact}-U_N\right\|_2\simeq\sqrt{h\sum_{j=1}^{N}\left|U_j^{exact}-(U_N)_j\right|^2},\tag{19}$$

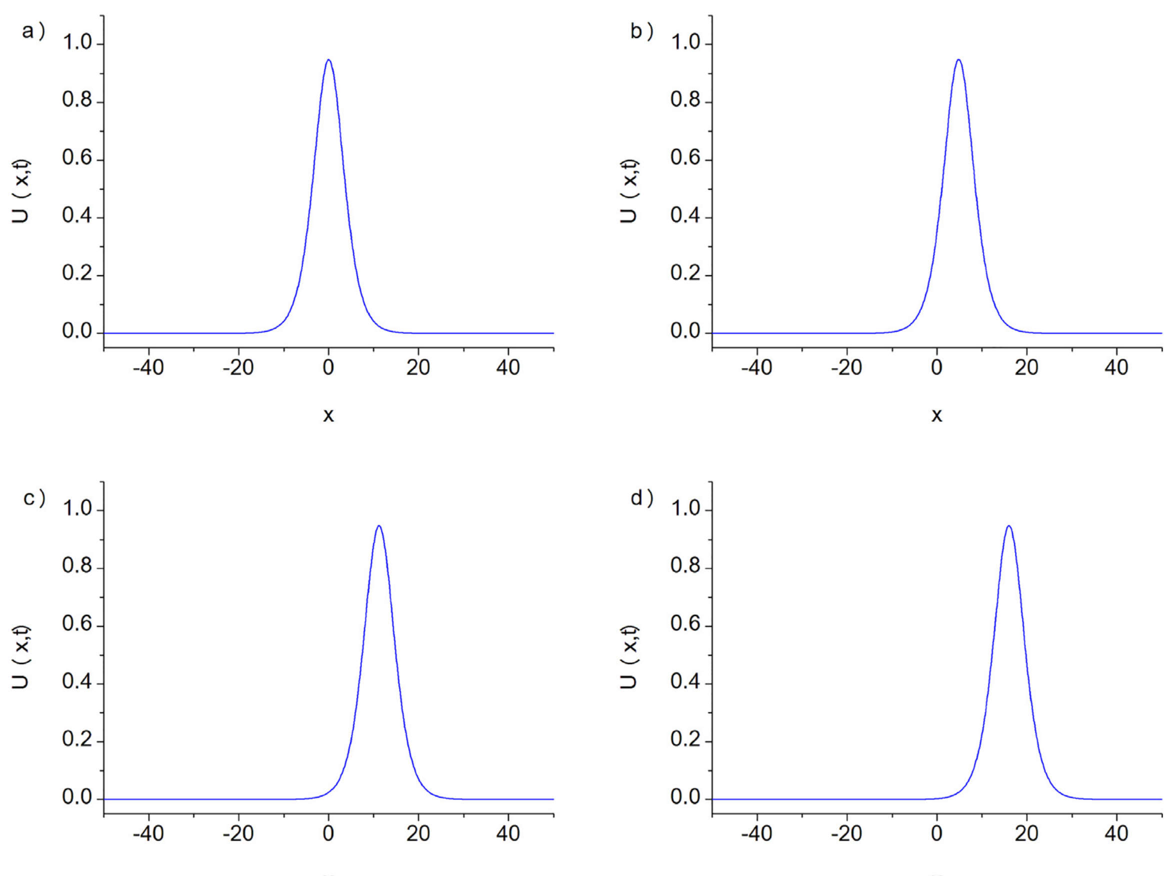

Fig. 2. Motion of the single solitary wave for $\alpha=\beta=\gamma=1$, $h=0.1$ and $\Delta t=0.05$ at a) $t=0$, b) $t=30$, c) $t=70$, d) $t=100$.

and the error norm L_∞

$$L_\infty = \left\| U^{exact} - U_N \right\|_\infty \simeq \max_j \left| U_j^{exact} - (U_N)_j \right|, \quad \times j = 1, 2, \ldots, N-1, \tag{20}$$

to calculate the difference between analytical and numerical solutions at some specified times. Kawahara Eq. (1) possesses only two conservation constants given by

$$I_1 = \int_a^b U dx \simeq h \sum_{j=1}^{N} U_j^n, \quad I_2 = \int_a^b U^2 dx \simeq h \sum_{j=1}^{N} (U_j^n)^2, \tag{21}$$

which correspond to conversation of mass and momentum, respectively [30]. In the simulation of solitary wave motion, the invariants I_1 and I_2 are monitored to check the conversation of the numerical algorithm.

5.1. The motion of single solitary wave

The solitary wave solution of the Kawahara Eq. (1) is given by considered with the boundary conditions $U \to 0$ as $x \to \pm\infty$ and the initial condition

$$U(x,t) = \frac{3}{\sqrt{10}} \sqrt{\frac{\beta^2}{\alpha\gamma}} \sec h^2 \left[\sqrt{\frac{1}{20}\frac{\beta}{\gamma}} \left(x - \frac{4}{25}\frac{\beta^2}{\gamma} t - x_0 \right) \right], \tag{22}$$

where $\alpha,\ \beta,\ \gamma$ and x_0 are arbitrary constants. The initial condition is

$$U(x,0) = \frac{3}{\sqrt{10}} \sqrt{\frac{\beta^2}{\alpha\gamma}} \sec h^2 \left[\sqrt{\frac{1}{20}\frac{\beta}{\gamma}} (x - x_0) \right], \tag{23}$$

where amplitude $A = \frac{3}{\sqrt{10}}\sqrt{\frac{\beta^2}{\alpha\gamma}}$ and velocity $c = \frac{4}{25}\frac{\beta^2}{\gamma}$.

For the numerical simulation of the motion of a single solitary wave, parameters $\alpha = \beta = \gamma = 1,\ h = 0.1$ and $\Delta t = 0.05$ over the interval $[-50, 50]$ are chosen. For these parameters, the solitary wave has an amplitude 0.94868. The conserved quantities and error norms L_2 and L_∞ are shown at selected times up to time $t = 100$. The obtained results are tabulated in Table 2. It can be seen from the Table 2 that the error norms L_2 and L_∞ are found to be small enough and the quantities in the invariants remain almost constant during the computer run. Percentage of relative changes of I_1 and I_2 are found to be 1.164×10^{-4} %, 6.854×10^{-10} %, respectively. In Fig. 1, the numerical solutions are displayed at $t = 0, 20, 40, \ldots, 100$. The graphs are plotted numerical solution of single soliton with $\alpha = \beta = \gamma = 1,\ h = 0.1$ and $\Delta t = 0.05$ at selected times from $t = 0$ to $t = 100$, in Fig. 2. Errors distributions at time $t = 100$ are depicted for solitary waves amplitudes 0.94868 in Fig. 3 to show the errors between the analytical and numerical results over the problem domain.

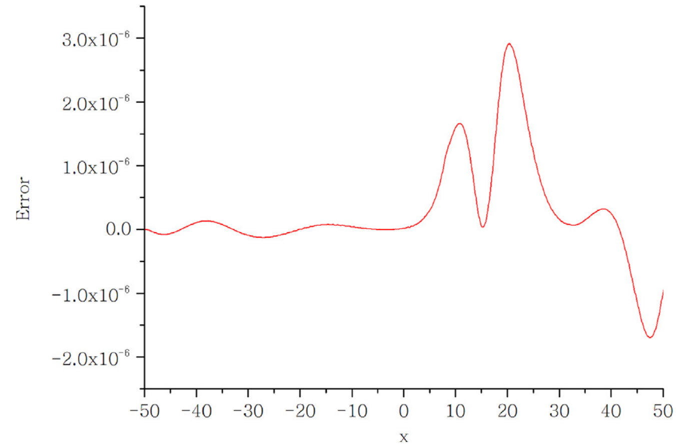

Fig. 3. Error for $\alpha = \beta = \gamma = 1,\ h = 0.1$ and $\Delta t = 0.05$, at $t = 100$.

Table 3
Comparison of invariants for the interaction of two solitary waves with $\alpha = \beta = \gamma = 1,\ h = 0.1,\ \Delta t = 0.05,\ c_1 = 0.85,\ c_2 = 0.35,\ x_1 = 0$ and $x_2 = 20$.

t	I_1	I_2
0	16.9705316311	20.3069218846
20	16.8465452348	20.1187647065
40	16.9748060046	20.0565572215
60	16.8992214653	19.9988890134
80	17.2676453891	19.9762119420
100	17.0413144553	19.9424422093

Table 4
Comparison of invariants for the interaction of three solitary waves with $\alpha = \beta = \gamma = 1,\ h = 0.1,\ \Delta t = 0.05,\ c_1 = 0.85,\ c_2 = 0.50,\ c_3 = 0.25,\ x_1 = -20,\ x_2 = 0$ and $x_3 = 20$.

t	I_1	I_2
0	25.4557818426	28.5673455186
25	25.6820777780	28.2475921308
50	25.7935768834	28.2118286972
75	25.1357840031	28.0967148921
100	24.9519313985	28.0933900647
125	24.9751678109	28.0875884719
150	25.0303969354	28.0810761856

Table 5
Invariants for Gaussian initial condition with $\alpha = \beta = \gamma = 1$ and $h = \Delta t = 0.25$ at $0 \le t \le 5$.

t	I_1	I_2
0	1.7724538509	1.2533141373
1	1.7724538507	1.2533141369
2	1.7724538506	1.2533141369
3	1.7724538506	1.2533141368
4	1.7724538506	1.2533141368
5	1.7724538506	1.2533141368

5.2. Interaction of two solitary waves

Secondly, we consider the interaction of two solitary waves by using the initial condition given by the linear sum of two well separated solitary waves having various

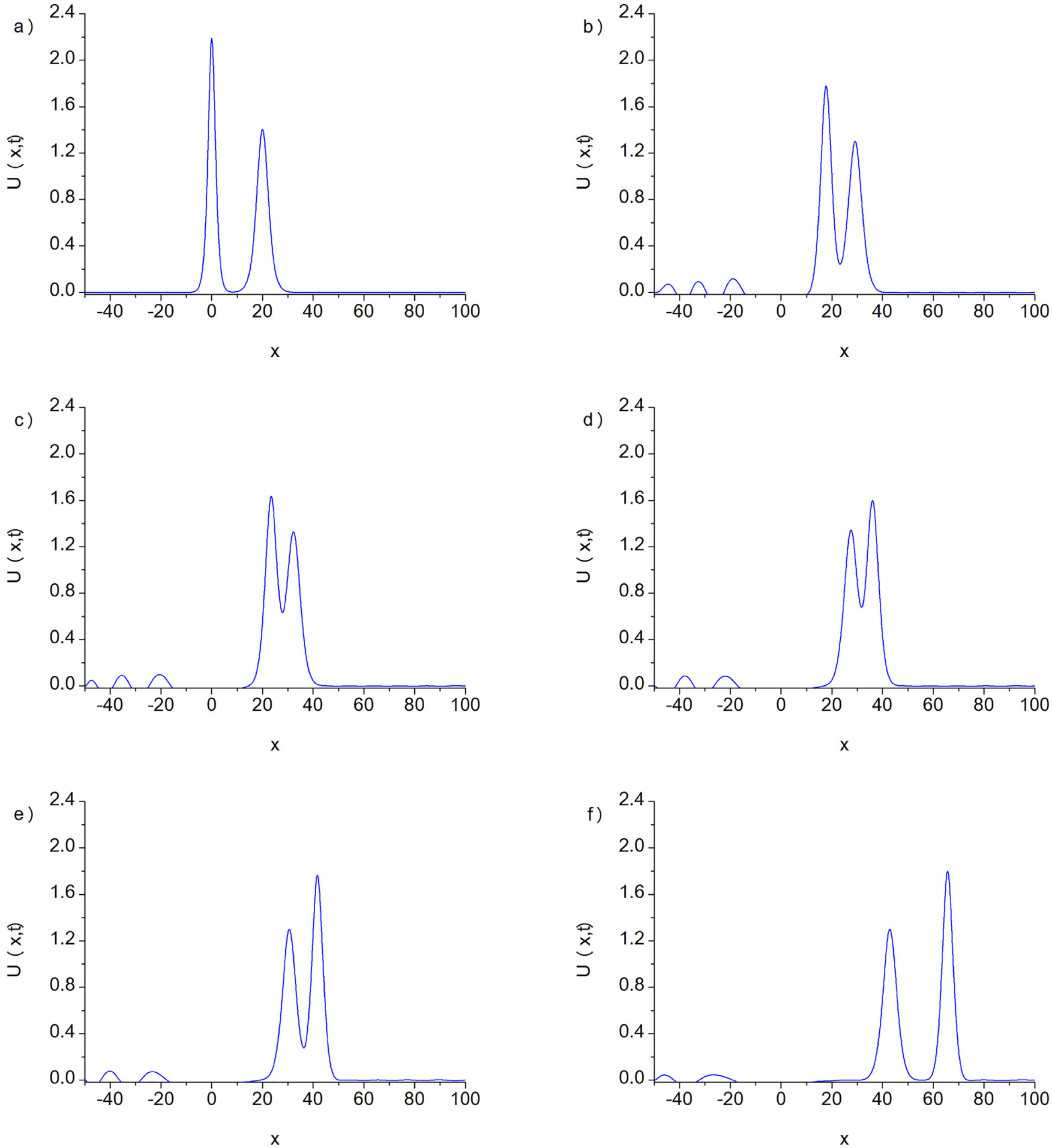

Fig. 4. Interaction of two solitary waves for $\alpha = \beta = \gamma = 1$, $h = 0.1$, $\Delta t = 0.05$, $c_1 = 0.85$, $c_2 = 0.35$, $x_1 = 0$ and $x_2 = 20$ at a) $t = 0$, b) $t = 30$, c) $t = 40$, d) $t = 50$, e) $t = 60$, f) $t = 100$.

amplitudes

$$U(x, 0) = \sum_{i=1}^{2} a_i \sec\mathrm{h}^2[b_i(x - x_i)], \tag{24}$$

where $a_i = \sqrt{\frac{45}{8}\frac{c_i}{\alpha}}$, $b_i = \sqrt{\frac{5}{16}\frac{c_i}{\beta}}$, $i = 1, 2$, c_i and x_i are arbitrary constants.

For the simulation, the parameters $\alpha = \beta = \gamma = 1$, $h = 0.1$, $\Delta t = 0.05$, $c_1 = 0.85$, $c_2 = 0.35$, $x_1 = 0$ and $x_2 = 20$ are chosen over the range $-50 \le x \le 100$. The experiment are run from $t = 0$ to $t = 100$ and the calculated values of the invariants I_1 and I_2 obtained by the present method are tabulated in Table 3. It is seen that the obtained values of the invariants remain almost constant during the computer run.

Fig. 4 shows the development of the interaction of two solitary waves. It is clear from the figure that, at $t = 0$ the greater solitary wave at the left position of the smaller solitary wave, at the beginning of the run. With the increases of the time the greater solitary wave catches up the smaller until at time $t = 30$, the smaller solitary wave being absorbed. The overlapping process continues until $t = 60$, greater soli-

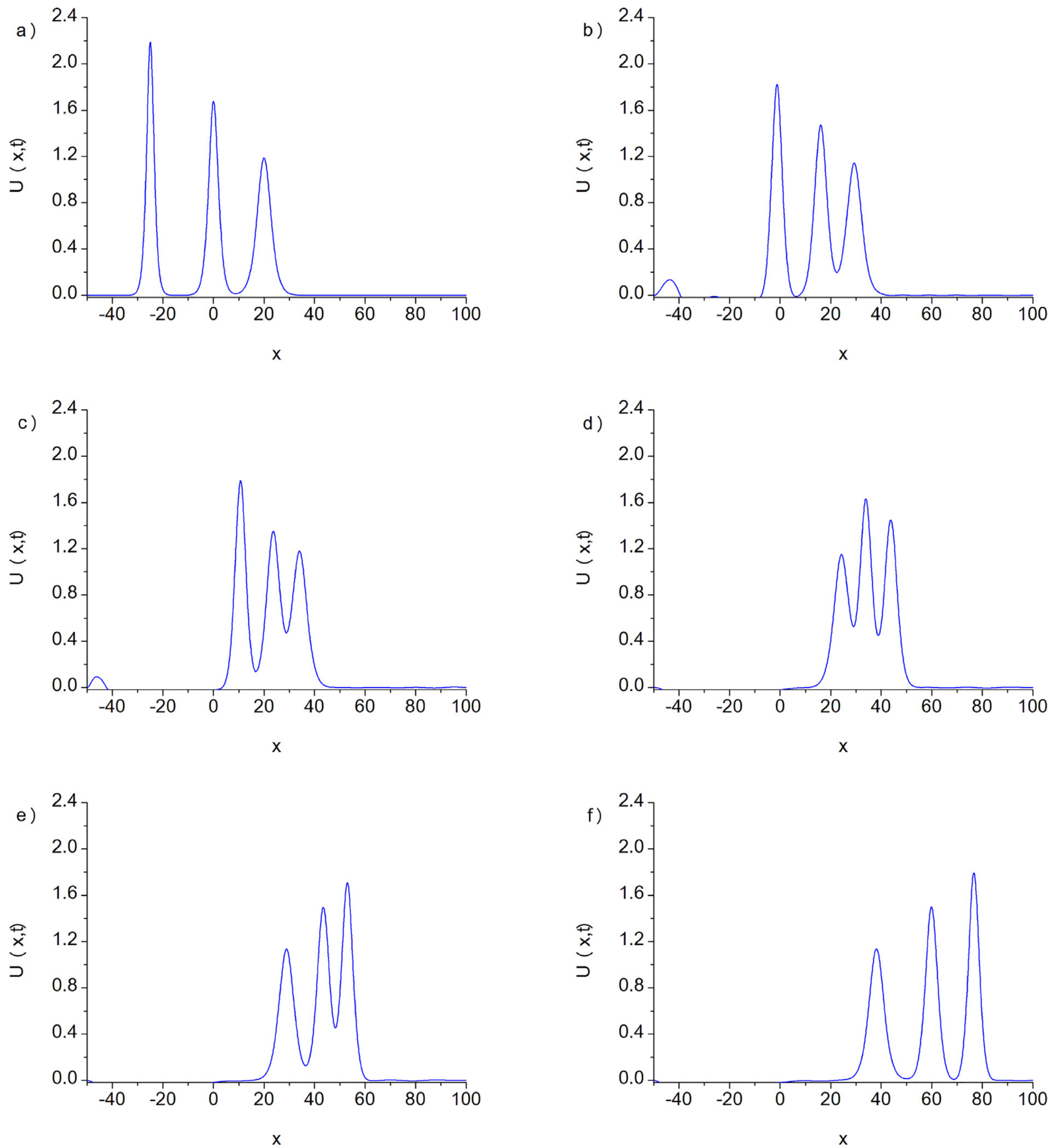

Fig. 5. Interaction of three solitary waves for $\alpha = \beta = \gamma = 1$, $h = 0.1$, $\Delta t = 0.05$, $c_1 = 0.85$, $c_2 = 0.50$, $c_3 = 0.25$, $x_1 = -25$, $x_2 = 0$ and $x_3 = 20$ at a) $t = 0$, b) $t = 40$, c) $t = 60$, d) $t = 90$,e) $t = 110$, f) $t = 150$.

tary wave has overtaken the smaller solitary wave and get in the process of the separating. At time $t = 100$, the interaction is complete and the greater solitary wave has separated completely.

5.3. Interaction of three solitary waves

Thirdly, we consider the interaction of three solitary waves by using the initial condition given by the linear sum of three well separated solitary waves having various amplitudes

$$U(x,0) = \sum_{i=1}^{3} a_i \sec\mathrm{h}^2[b_i(x - x_i)], \tag{25}$$

where $a_i = \sqrt{\frac{45}{8}\frac{c_i}{\alpha}}$, $b_i = \sqrt{\frac{5}{16}\frac{c_i}{\beta}}$, $i = 1, 2, 3$, c_i and x_i are arbitrary constants.

For the computational work, the parameters $\alpha = \beta = \gamma = 1$, $h = 0.1$, $\Delta t = 0.05$, $c_1 = 0.85$, $c_2 = 0.50$, $c_3 = 0.25$, $x_1 = -25$, $x_2 = 0$ and $x_3 = 20$ are taken over

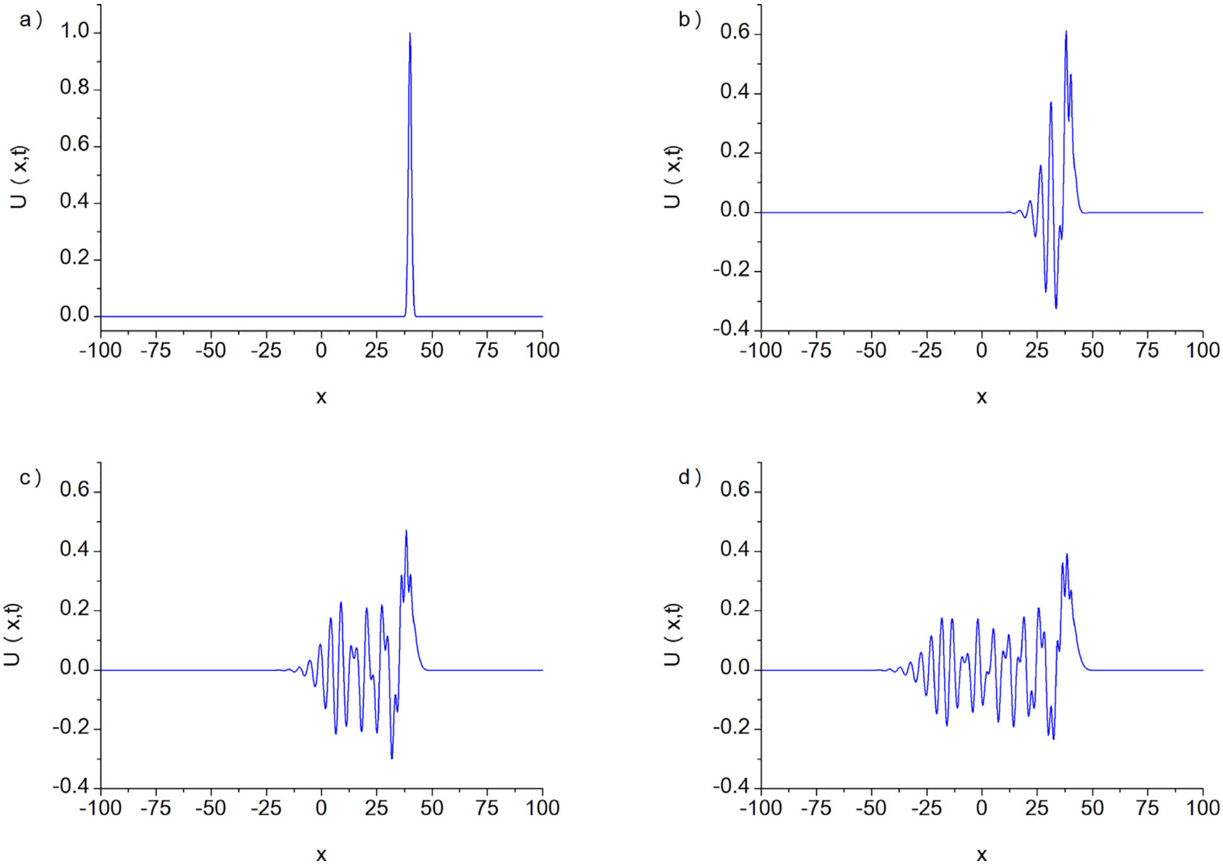

Fig. 6. Evolution of waves for $\alpha = \beta = \gamma = 1$ and $h = \Delta t = 0.25$ at a) $t = 0$, b) $t = 1$, c) $t = 3$, d) $t = 5$.

the range $-50 \leq x \leq 100$. Simulations are done up to time $t = 150$. Table 4 displays values of the conserved quantities pending the travelling. It is seen from the Table 4 that the obtained values of the invariants remain almost during the computer run. In Fig. 5, the interaction of three solitary waves is depicted. As it is seen from the Fig. 5, interaction started about time $t = 40$, overlapping processes occurred between time $t = 40$ and $t = 110$ and waves started to resume their original shapes after the time $t = 150$.

6. Evolution of solitons

Evolution of a train of solitons of the Kawahara equation has been studied using the Gaussian initial condition

$$U(x, 0) = \exp\left[-(x - 40)^2\right]. \tag{26}$$

The values of $\alpha = \beta = \gamma = 1$, $h = 0.25$ and $\Delta t = 0.25$ are chosen at the region of the $-100 \leq x \leq 100$. The numerical computations are done up to $t = 5$. The values of the two invariants of motion are presented in Table 5. Percentage of relative changes of I_1 and I_2 are found to be 1.782×10^{-8} %, 4.048×10^{-8} %, respectively. As seen in the Fig. 6, evolution of any soliton doesn't occur with Gaussian initial condition. However, as time progressed it is observed that the waves made oscillation increasingly.

7. Conclusion

In this paper, we have obtained the solitary wave solutions of the modified Kawahara equation by using collocation method based on septic B-spline functions. To prove the performance of numerical scheme, the error norms L_2, L_∞ for single solitary wave and two invariants I_1 and I_2 for three test problems have been calculated. It has been observed that the error norms are satisfactorily small and the invariants are well conserved. Also, the linearized numerical scheme is unconditionally stable. The method successfully models the motion and interaction of the solitary waves and evolution of solitons. Finally, we can say that this method can a reliable method for obtaining the numerical solutions of similar type non-linear equations.

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