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Original Research Paper

Quantum Calculus Approach to the Dual Bicomplex Fibonacci and Lucas Numbers

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Abstract. Quantum calculus, which arises in the mathematical fields of combinatorics and special functions as well as in a number of areas, involving the study of fractals and multi−fractal measures, and expressions for the entropy of chaotic dynamical systems, has attracted the attention of many researchers in recent years. In this paper, by virtue of some useful notations from $q-$calculus, we define the $q-$Fibonacci dual bicomplex numbers and $q-$Lucas dual bicomplex numbers with a different perspective. Afterwards, we give the Binet formulas, binomial sums, exponential generating functions, Catalan identities, Cassini identities, d'Ocagne identities and some algebraic properties for the $q-$Fibonacci dual bicomplex numbers and $q-$Lucas dual bicomplex numbers.

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1 Introduction

The Italian mathematician Gerolamo Cardano first discovered complex numbers while trying to solve a simpler state of the cubic equation. After, Leonard Euler illustrated the complex numbers as points with rectangular coordinates by using the notation $i = \sqrt{-1}$. In 19th century, Clifford presented the dual number system and dual numbers as in the form $A = a + \varepsilon a^*$, where $a, a^* \in \mathbb{R}$, $\varepsilon \neq 0$ and $\varepsilon^2 = 0$ [6]. Dual numbers and dual complex numbers emerge in many areas in physics and mathematics such as coordinate transformation, matrix modeling, displacement analysis, rigid body dynamics, velocity analysis, static analysis, dynamic analysis, 2D rigid transformation, mechanics, kinematics and applications of geometry. So far, there are number of studies in the literature related with dual numbers and dual complex numbers [2, 5, 8, 17, 19, 20, 27]. A dual complex number w is an ordered pair of complex numbers (z, t) associated with the complex unit 1 and dual unit ε which is a nilpotent number such that $\varepsilon \neq 0$ and $\varepsilon^2 = 0$. Messelmi, in [20], defined the set of dual complex numbers as

$$\mathbb{DC} = \{w = z + \varepsilon t \mid (z, t) \in \mathbb{C}, \varepsilon \neq 0 \quad \text{and} \quad \varepsilon^2 = 0\}. \tag{1}$$

He also studied generalization of the concept of holomorphicity to dual complex functions using complex analysis. In (1), if $z = x_1 + \mathbf{i}x_2$ and $t = y_1 + \mathbf{i}y_2$, then any dual complex number can be expressed by $w = x_1 + \mathbf{i}x_2 + \varepsilon y_1 + \varepsilon \mathbf{i} y_2$. Another interesting topic is bicomplex numbers which arise in various areas such as quantum mechanics, digital signal processing, electromagnetic waves and curved structures, determination of antenna patterns, fractal structures and many related fields of physics and mathematics [18, 23–26]. Any set of bicomplex numbers can be expressed by

$$\mathbb{C}_2 = \{q_1 + \mathbf{i}q_2 + \mathbf{j}q_3 + \mathbf{ij}q_4 \mid q_1, q_2, q_3, q_4 \in \mathbb{R}\}, \tag{2}$$

where the basis $1, \mathbf{i}, \mathbf{j}$ and $\mathbf{ij}$ satisfy the conditions $\mathbf{i}^2 = -1$, $\mathbf{j}^2 = -1$ and $\mathbf{ij} = \mathbf{ji}$ [21]. By taking into account the definition of the dual numbers and bicomplex numbers the dual bicomplex numbers are defined by

$$\tilde{X} = X + \varepsilon X^*, \tag{3}$$

where $X = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{ij}x_3$ and $X^* = x_0^* + \mathbf{i}x_1^* + \mathbf{j}x_2^* + \mathbf{ij}x_3^*$ [3]. Let $\tilde{X}$ and $\tilde{Y}$ be two dual bicomplex numbers. Then, the addition, subtraction and multiplication of two dual bicomplex numbers are defined by

$$\tilde{X} \pm \tilde{Y} = (X \pm Y) + \varepsilon (X^* \pm Y^*), \tag{4}$$

$$\tilde{X}.\tilde{Y} = XY + \varepsilon (YX^* + XY^*), \tag{5}$$

respectively. Furthermore, three different conjugations of dual bicomplex numbers, according to the imaginary units $\mathbf{i}, \mathbf{j}$ and $\mathbf{ij}$, can be expressed by

$$\overline{\tilde{X}} = (x_0 + \varepsilon x_0^*) - (x_1 + \varepsilon x_1^*)\,\mathbf{i} + (x_2 + \varepsilon x_2^*)\,\mathbf{j} - (x_3 + \varepsilon x_3^*)\,\mathbf{ij}, \tag{6}$$

$$\overline{\tilde{X}} = (x_0 + \varepsilon x_0^*) + (x_1 + \varepsilon x_1^*)\,\mathbf{i} - (x_2 + \varepsilon x_2^*)\,\mathbf{j} - (x_3 + \varepsilon x_3^*)\,\mathbf{ij}, \tag{7}$$

$$\overline{\tilde{X}} = (x_0 + \varepsilon x_0^*) - (x_1 + \varepsilon x_1^*)\,\mathbf{i} - (x_2 + \varepsilon x_2^*)\,\mathbf{j} + (x_3 + \varepsilon x_3^*)\,\mathbf{ij} \tag{8}$$

The Fibonacci and Lucas numbers play an important role in various areas such as mathematics, physics, computer science and related fields. For more information about Fibonacci and Lucas numbers and their properties, we refer to book [16]. For $n \in \mathbb{N}_0$, the Fibonacci and Lucas numbers are defined by the recurrence relations

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \tag{9}$$

and

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1, \tag{10}$$

respectively. The Binet formulas for the Fibonacci and Lucas numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n, \tag{11}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic polynomial $x^2 - x - 1 = 0$. Until now, few researchers have studied the bicomplex numbers with Fibonacci and Lucas numbers [9, 10, 21, 28]. For example, Nurkan and Güven defined the bicomplex Fibonacci numbers and bicomplex Lucas numbers and they examined the algebraic properties of these numbers [21]. Halıcı and Karataş defined the bicomplex Horadam numbers and they gave some additional identities for these numbers [10]. Moreover, they obtained the Binet formula and generating functions for these numbers. Motivated by the above cited works, Babadag, in [3], defined the dual bicomplex Fibonacci and dual bicomplex Lucas numbers as

$$\tilde{x}_n = F_n + \mathbf{i}F_{n+1} + \mathbf{j}F_{n+2} + \mathbf{ij}F_{n+3} + \varepsilon\left(F_{n+1} + \mathbf{i}F_{n+2} + \mathbf{j}F_{n+3} + \mathbf{ij}F_{n+4}\right) \quad (12)$$

and

$$\tilde{k}_n = L_n + \mathbf{i}L_{n+1} + \mathbf{j}L_{n+2} + \mathbf{ij}L_{n+3} + \varepsilon\left(L_{n+1} + \mathbf{i}L_{n+2} + \mathbf{j}L_{n+3} + \mathbf{ij}L_{n+4}\right), \quad (13)$$

where F_n and L_n are the $n-$th Fibonacci and Lucas numbers respectively. In addition, in (12) and (13), $\mathbf{i}, \mathbf{j}$ and $\mathbf{ij}$ are the imaginary units and ε is the dual unit which satisfy the conditions $\mathbf{i}^2 = -1, \mathbf{j}^2 = -1, \mathbf{ij} = \mathbf{ji}$ and $\varepsilon^2 = 0$.

Quantum calculus, which may be viewed as generalization of ordinary calculus, plays an important role in physics, combinatorics, number theory and other fields of the mathematics. As there is a relationship between quantum calculus and number sequences, this study can be extended to different areas by defining a relationship between fractal calculus and fractional calculus and quantum calculus. For more information on fractal and fractional calculus, we refer the readers to [11, 13, 14, 22]. In the second half of the twentieth century, studies related with quantum calculus and its applications to mathematics and physics have increased significantly. Up to the present, with the help of $q-$calculus, some researchers have investigated the properties of quaternions and hybrid numbers using Fibonacci and Lucas numbers [1, 15].

Now, we give definitions and facts from the quantum calculus necessary for understanding of this paper. For $n \in \mathbb{N}_0$, a $q-$integer is defined by

$$[n]_q = \frac{1-q^n}{1-q} = q^{n-1} + \ldots + q + 1. \quad (14)$$

By means of (14), for $m, n \in \mathbb{Z}$, we get

$$[m+n]_q = [m]_q + q^m [n]_q. \quad (15)$$

For more information about quantum calculus, we refer to the book [12] to the readers.

By analogy to the earlier works, in this paper, with a different perspective, we define the $q-$Fibonacci dual bicomplex numbers and $q-$Lucas dual bicomplex numbers by using the notations from quantum calculus. We obtain several new results which are the generalization of different dual bicomplex numbers.

2 $q-$Fibonacci Dual Bicomplex Numbers and $q-$Lucas Dual Bicomplex Numbers

In this section, we define a new generalization of the dual bicomplex Fibonacci numbers and dual bicomplex Lucas numbers. Moreover, with the help of $q-$integer, we obtain the Binet formulas, exponential generating functions, several binomial sum identities, Catalan identity, Cassini identity and d'Ocagne identity for the $q-$Fibonacci dual bicomplex numbers and $q-$Lucas dual bicomplex numbers.

Definition 2.1. *The $q-$Fibonacci and $q-$Lucas dual bicomplex numbers are defined by*

$$\begin{aligned}\widetilde{\mathcal{F}}_n(\alpha;q) &= \alpha^{n-1}[n]_q + \alpha^n[n+1]_q\mathbf{i} + \alpha^{n+1}[n+2]_q\mathbf{j} + \alpha^{n+2}[n+3]_q\mathbf{ij} \\ &\quad + \varepsilon\left(\alpha^n[n+1]_q + \alpha^{n+1}[n+2]_q\mathbf{i} + \alpha^{n+2}[n+3]_q\mathbf{j} + \alpha^{n+3}[n+4]_q\mathbf{ij}\right) \quad (16)\end{aligned}$$

and

$$\begin{aligned}&\widetilde{\mathcal{L}}_n(\alpha;q) \\ &= \alpha^n\frac{[2n]_q}{[n]_q} + \alpha^{n+1}\frac{[2n+2]_q}{[n+1]_q}\mathbf{i} + \alpha^{n+2}\frac{[2n+4]_q}{[n+2]_q}\mathbf{j} + \alpha^{n+3}\frac{[2n+6]_q}{[n+3]_q}\mathbf{ij} \\ &+ \varepsilon\left(\alpha^{n+1}\frac{[2n+2]_q}{[n+1]_q} + \alpha^{n+2}\frac{[2n+4]_q}{[n+2]_q}\mathbf{i} + \alpha^{n+3}\frac{[2n+6]_q}{[n+3]_q}\mathbf{j} + \alpha^{n+4}\frac{[2n+8]_q}{[n+4]_q}\mathbf{ij}\right). \quad (17)\end{aligned}$$

It is not difficult to see that the $q-$Fibonacci dual bicomplex numbers and the $q-$Lucas dual bicomplex numbers can be reduced to several dual bicomplex numbers for the special cases of q and α. For example,

- if we get $\alpha = \frac{1+\sqrt{5}}{2}$ and $q = -\frac{1}{\alpha^2}$ in (16), we obtain Dual bicomplex Fibonacci numbers,
- if we get $\alpha = 1+\sqrt{2}$ and $q = -\frac{1}{\alpha^2}$ in (16), we obtain Dual bicomplex Pell numbers,

- if we get $\alpha = \frac{k+\sqrt{k^2+4}}{2}$ and $q = -\frac{1}{\alpha^2}$ in (16), we obtain Dual bicomplex $k-$Fibonacci numbers,
- if we get $\alpha = 2$ and $q = -\frac{1}{2}$ in (16), we obtain Dual bicomplex Jacobsthal numbers,
- if we get $1+\sqrt{1+k}$ and $q = -\frac{k}{\alpha^2}$ in (16), we obtain Dual bicomplex $k-$Pell numbers,
- if we get $\alpha = \frac{1+\sqrt{5}}{2}$ and $q = -\frac{1}{\alpha^2}$ in (17), we obtain Dual bicomplex Lucas numbers,
- if we get $\alpha = 1+\sqrt{2}$ and $q = -\frac{1}{\alpha^2}$ in (17), we obtain Dual bicomplex Pell$-$Lucas numbers,
- if we get $\alpha = \frac{k+\sqrt{k^2+4}}{2}$ and $q = -\frac{1}{\alpha^2}$ in (17), we obtain Dual bicomplex $k-$Lucas numbers,
- if we get $\alpha = 2$ and $q = -\frac{1}{2}$ in (17), we obtain Dual bicomplex Jacobsthal$-$Lucas numbers,
- if we get $1+\sqrt{1+k}$ and $q = -\frac{k}{\alpha^2}$ in (17), we obtain Dual bicomplex $k-$Pell$-$Lucas numbers.

The Binet formula was derived by Binet in 1843, although the result was known to Euler, Daniel Bernoulli, and de Moivre more than a century earlier. So, the following theorem describes the $q-$analog of the Binet formula of the Fibonacci and Lucas dual bicomplex numbers.

Theorem 2.2. *The Binet formula for the $q-$Fibonacci dual bicomplex numbers and $q-$Lucas dual bicomplex numbers are*

$$\widetilde{\mathcal{F}}_n(\alpha;q) = \frac{\alpha^n \underline{\alpha} - (\alpha q)^n \underline{\gamma}}{\alpha(1-q)} + \varepsilon \left(\frac{\alpha^{n+1} \underline{\alpha} - (\alpha q)^{n+1} \underline{\gamma}}{\alpha(1-q)} \right) \tag{18}$$

and

$$\widetilde{\mathcal{L}}_n(\alpha;q) = \alpha^n \underline{\alpha} + (\alpha q)^n \underline{\gamma} + \varepsilon \left(\alpha^{n+1} \underline{\alpha} + (\alpha q)^{n+1} \underline{\gamma} \right), \tag{19}$$

where $\underline{\alpha} = \left(1 + \boldsymbol{i}\alpha + \boldsymbol{j}\alpha^2 + \boldsymbol{ij}\alpha^3\right)$ *and* $\underline{\gamma} = \left(1 + \boldsymbol{i}(\alpha q) + \boldsymbol{j}(\alpha q)^2 + \boldsymbol{ij}(\alpha q)^3\right)$.

Proof. From (14) and (16), we find that

$$
\begin{aligned}
&\widetilde{\mathcal{F}}_n(\alpha;q)\\
&=\alpha^{n-1}\frac{1-q^n}{1-q}+\alpha^n\frac{1-q^{n+1}}{1-q}\mathbf{i}+\alpha^{n+1}\frac{1-q^{n+2}}{1-q}\mathbf{j}+\alpha^{n+2}\frac{1-q^{n+3}}{1-q}\mathbf{ij}\\
&\quad+\varepsilon\left(\alpha^n\frac{1-q^{n+1}}{1-q}+\alpha^{n+1}\frac{1-q^{n+2}}{1-q}\mathbf{i}+\alpha^{n+2}\frac{1-q^{n+3}}{1-q}\mathbf{j}+\alpha^{n+3}\frac{1-q^{n+4}}{1-q}\mathbf{ij}\right)\\
&=\frac{\alpha^n}{\alpha(1-q)}\left(1+\mathbf{i}\alpha+\mathbf{j}\alpha^2+\mathbf{ij}\alpha^3\right)-\frac{(\alpha q)^n}{\alpha(1-q)}\left(1+\mathbf{i}(\alpha q)+\mathbf{j}(\alpha q)^2+\mathbf{ij}(\alpha q)^3\right)\\
&\quad+\varepsilon\left(\frac{\alpha^{n+1}}{\alpha(1-q)}\left(1+\mathbf{i}\alpha+\mathbf{j}\alpha^2+\mathbf{ij}\alpha^3\right)+\frac{(\alpha q)^{n+1}}{\alpha(1-q)}\left(1+\mathbf{i}(\alpha q)+\mathbf{j}(\alpha q)^2+\mathbf{ij}(\alpha q)^3\right)\right)\\
&=\frac{\alpha^n\underline{\alpha}-(\alpha q)^n\underline{\gamma}}{\alpha(1-q)}+\varepsilon\left(\frac{\alpha^{n+1}\underline{\alpha}-(\alpha q)^{n+1}\underline{\gamma}}{\alpha(1-q)}\right).
\end{aligned}
$$

On the other hand, from (14) and (17), we get

$$
\begin{aligned}
&\widetilde{\mathcal{L}}_n(\alpha;q)\\
&=\alpha^n\frac{1-q^{2n}}{1-q^n}+\alpha^{n+1}\frac{1-q^{2n+2}}{1-q^{n+1}}\mathbf{i}+\alpha^{n+2}\frac{1-q^{2n+4}}{1-q^{n+2}}\mathbf{j}+\alpha^{n+3}\frac{1-q^{2n+6}}{1-q^{n+3}}\mathbf{ij}\\
&\quad+\varepsilon\left(\alpha^{n+1}\frac{1-q^{2n+2}}{1-q^{n+1}}+\alpha^{n+2}\frac{1-q^{2n+4}}{1-q^{n+2}}\mathbf{i}+\alpha^{n+3}\frac{1-q^{2n+6}}{1-q^{n+3}}\mathbf{j}+\alpha^{n+4}\frac{1-q^{2n+8}}{1-q^{n+4}}\mathbf{ij}\right)\\
&=\alpha^n\left(1+\mathbf{i}\alpha+\mathbf{j}\alpha^2+\mathbf{ij}\alpha^3\right)+(\alpha q)^n\left(1+\mathbf{i}(\alpha q)+\mathbf{j}(\alpha q)^2+\mathbf{ij}(\alpha q)^3\right)\\
&\quad+\varepsilon\left(\alpha^{n+1}\left(1+\mathbf{i}\alpha+\mathbf{j}\alpha^2+\mathbf{ij}\alpha^3\right)+(\alpha q)^{n+1}\left(1+\mathbf{i}(\alpha q)+\mathbf{j}(\alpha q)^2+\mathbf{ij}(\alpha q)^3\right)\right)\\
&=\alpha^n\underline{\alpha}+(\alpha q)^n\underline{\gamma}+\varepsilon\left(\alpha^{n+1}\underline{\alpha}+(\alpha q)^{n+1}\underline{\gamma}\right),
\end{aligned}
$$

which is the desired result. $\square$

Theorem 2.3. *The exponential generating functions for the $q-$Fibonacci dual bicomplex numbers and $q-$Lucas dual bicomplex numbers are*

$$\sum_{n=0}^{\infty}\widetilde{\mathcal{F}}_n(\alpha;q)\frac{x^n}{n!}=\frac{e^{\alpha x}\underline{\alpha}-e^{(\alpha q)x}\underline{\gamma}}{\alpha(1-q)}+\varepsilon\left(\frac{e^{\alpha x}\alpha\underline{\alpha}-e^{(\alpha q)x}\alpha q\underline{\gamma}}{\alpha(1-q)}\right)\tag{20}$$

and

$$\sum_{n=0}^{\infty}\widetilde{\mathcal{L}}_n(\alpha;q)\frac{x^n}{n!}=e^{\alpha x}\underline{\alpha}+e^{(\alpha q)x}\underline{\gamma}+\varepsilon\left(e^{\alpha x}\alpha\underline{\alpha}+e^{(\alpha q)x}\alpha q\underline{\gamma}\right),\tag{21}$$

respectively.

Proof. Using the Binet formula of the $q-$Fibonacci dual bicomplex numbers, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\widetilde{\mathcal{F}}_n(\alpha;q)\frac{x^n}{n!} &= \sum_{n=0}^{\infty}\left[\frac{\alpha^n\underline{\alpha}-(\alpha q)^n\underline{\gamma}}{\alpha(1-q)}+\varepsilon\left(\frac{\alpha^{n+1}\underline{\alpha}-(\alpha q)^{n+1}\underline{\gamma}}{\alpha(1-q)}\right)\right]\frac{x^n}{n!}\\
&=\frac{\underline{\alpha}}{\alpha(1-q)}\sum_{n=0}^{\infty}\alpha^n\frac{x^n}{n!}-\frac{\underline{\gamma}}{\alpha(1-q)}\sum_{n=0}^{\infty}(\alpha q)^n\frac{x^n}{n!}\\
&\quad+\varepsilon\left(\frac{\underline{\alpha}\alpha}{\alpha(1-q)}\sum_{n=0}^{\infty}\alpha^n\frac{x^n}{n!}-\frac{\underline{\gamma}\alpha q}{\alpha(1-q)}\sum_{n=0}^{\infty}(\alpha q)^n\frac{x^n}{n!}\right)\\
&=\frac{e^{\alpha x}\underline{\alpha}-e^{(\alpha q)x}\underline{\gamma}}{\alpha(1-q)}+\varepsilon\left(\frac{e^{\alpha x}\alpha\underline{\alpha}-e^{(\alpha q)x}\alpha q\underline{\gamma}}{\alpha(1-q)}\right). \qquad (22)
\end{aligned}$$

Moreover, from (19), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\widetilde{\mathcal{L}}_n(\alpha;q)\frac{x^n}{n!} &= \sum_{n=0}^{\infty}\left[\alpha^n\underline{\alpha}+(\alpha q)^n\underline{\gamma}+\varepsilon\left(\alpha^{n+1}\underline{\alpha}+(\alpha q)^{n+1}\underline{\gamma}\right)\right]\frac{x^n}{n!}\\
&=\underline{\alpha}\sum_{n=0}^{\infty}\alpha^n\frac{x^n}{n!}+\underline{\gamma}\sum_{n=0}^{\infty}(\alpha q)^n\frac{x^n}{n!}\\
&\quad+\varepsilon\left(\underline{\alpha}\alpha\sum_{n=0}^{\infty}\alpha^n\frac{x^n}{n!}+\underline{\gamma}\alpha q\sum_{n=0}^{\infty}(\alpha q)^n\frac{x^n}{n!}\right)\\
&=e^{\alpha x}\underline{\alpha}+e^{(\alpha q)x}\underline{\gamma}+\varepsilon\left(e^{\alpha x}\alpha\underline{\alpha}+e^{(\alpha q)x}\alpha q\underline{\gamma}\right). \qquad (23)
\end{aligned}$$

Thus, the proof is completed. □

Theorem 2.4. *For nonnegative integers n and j, the following identities hold:*

$$\sum_{k=0}^{n}\binom{n}{k}\left(-\alpha^2 q\right)^{n-k}\widetilde{\mathcal{F}}_{2k+j}(\alpha;q)=\begin{cases}(\alpha-\alpha q)^n\,\widetilde{\mathcal{F}}_{n+j}(\alpha;q), & \text{if } n \text{ is even}\\ (\alpha-\alpha q)^{n-1}\,\widetilde{\mathcal{L}}_{n+j}(\alpha;q), & \text{if } n \text{ is odd}\end{cases}, \qquad (24)$$

$$\sum_{k=0}^{n}\binom{n}{k}\left(-\alpha^2 q\right)^{n-k}\widetilde{\mathcal{L}}_{2k+j}(\alpha;q)=\begin{cases}(\alpha-\alpha q)^n\,\widetilde{\mathcal{L}}_{n+j}(\alpha;q), & \text{if } n \text{ is even}\\ (\alpha-\alpha q)^{n+1}\,\widetilde{\mathcal{F}}_{n+j}(\alpha;q), & \text{if } n \text{ is odd}\end{cases}, \qquad (25)$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(-\alpha^2 q\right)^{n-k}\widetilde{\mathcal{F}}_{2k+j}(\alpha;q)=\left(-\alpha\,[2]_q\right)^n\widetilde{\mathcal{F}}_{n+j}(\alpha;q), \tag{26}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(-\alpha^2 q\right)^{n-k}\widetilde{\mathcal{L}}_{2k+j}(\alpha;q)=\left(-\alpha\,[2]_q\right)^n\widetilde{\mathcal{L}}_{n+j}(\alpha;q). \tag{27}$$

Proof. We first prove the identity (24). By means of (18), we get

$$\sum_{k=0}^{n}\binom{n}{k}\left(-\alpha^2 q\right)^{n-k}\widetilde{\mathcal{F}}_{2k+j}(\alpha;q)$$

$$\begin{aligned}
&=\sum_{k=0}^{n}\binom{n}{k}\left(-\alpha^2 q\right)^{n-k}\left[\frac{\alpha^{2k+j}\underline{\alpha}-(\alpha q)^{2k+j}\underline{\gamma}}{\alpha(1-q)}+\varepsilon\left(\frac{\alpha^{2k+j+1}\underline{\alpha}-(\alpha q)^{2k+j+1}\underline{\gamma}}{\alpha(1-q)}\right)\right]\\
&=\frac{\alpha^j\underline{\alpha}}{\alpha(1-q)}\sum_{k=0}^{n}\binom{n}{k}\left(-\alpha^2 q\right)^{n-k}\alpha^{2k}-\frac{(\alpha q)^j\underline{\gamma}}{\alpha(1-q)}\sum_{k=0}^{n}\binom{n}{k}\left(-\alpha^2 q\right)^{n-k}(\alpha q)^{2k}\\
&\quad+\varepsilon\left(\frac{\alpha^{j+1}\underline{\alpha}}{\alpha(1-q)}\sum_{k=0}^{n}\binom{n}{k}\left(-\alpha^2 q\right)^{n-k}\alpha^{2k}-\frac{(\alpha q)^{j+1}\underline{\gamma}}{\alpha(1-q)}\sum_{k=0}^{n}\binom{n}{k}\left(-\alpha^2 q\right)^{n-k}(\alpha q)^{2k}\right)\\
&=\frac{\alpha^j\underline{\alpha}\left(\alpha^2-\alpha^2 q\right)^n-(\alpha q)^j\underline{\gamma}\left((\alpha q)^2-\alpha^2 q\right)^n}{\alpha(1-q)}\\
&\quad+\varepsilon\left(\frac{\alpha^{j+1}\underline{\alpha}\left(\alpha^2-\alpha^2 q\right)^n-(\alpha q)^{j+1}\underline{\gamma}\left((\alpha q)^2-\alpha^2 q\right)^n}{\alpha(1-q)}\right)\\
&=\frac{\left(\alpha\left(\alpha-\alpha q\right)\right)^n\alpha^j\underline{\alpha}-\left(-\alpha q\left(\alpha-\alpha q\right)\right)^n(\alpha q)^j\underline{\gamma}}{\alpha(1-q)}\\
&\quad+\varepsilon\left(\frac{\left(\alpha\left(\alpha-\alpha q\right)\right)^n\alpha^{j+1}\underline{\alpha}-\left(-\alpha q\left(\alpha-\alpha q\right)\right)^n(\alpha q)^{j+1}\underline{\gamma}}{\alpha(1-q)}\right).
\end{aligned}\tag{28}$$

If n is even in (28), we have

$$\sum_{k=0}^{n}\binom{n}{k}\left(-\alpha^2 q\right)^{n-k}\widetilde{\mathcal{F}}_{2k+j}(\alpha;q)$$

$$= \frac{(\alpha(\alpha-\alpha q))^n \alpha^j \underline{\alpha} - (\alpha q(\alpha-\alpha q))^n (\alpha q)^j \underline{\gamma}}{\alpha(1-q)} + \varepsilon\left(\frac{(\alpha(\alpha-\alpha q))^n \alpha^{j+1}\underline{\alpha} - (\alpha q(\alpha-\alpha q))^n (\alpha q)^{j+1}\underline{\gamma}}{\alpha(1-q)}\right)$$
$$= (\alpha-\alpha q)^n\left[\frac{\alpha^{n+j}\underline{\alpha} - (\alpha q)^{n+j}\underline{\gamma}}{\alpha(1-q)} + \varepsilon\left(\frac{\alpha^{n+j+1}\underline{\alpha} - (\alpha q)^{n+j+1}\underline{\gamma}}{\alpha(1-q)}\right)\right]$$
$$= (\alpha-\alpha q)^n \widetilde{\mathcal{F}}_{n+j}(\alpha;q).$$

On the other hand, if n is odd in (28), we obtain

$$\sum_{k=0}^{n}\binom{n}{k}\left(-\alpha^2 q\right)^{n-k}\widetilde{\mathcal{F}}_{2k+j}(\alpha;q)$$

$$= \frac{(\alpha(\alpha-\alpha q))^n \alpha^j \underline{\alpha} + (\alpha q(\alpha-\alpha q))^n (\alpha q)^j \underline{\gamma}}{\alpha(1-q)} + \varepsilon\left(\frac{(\alpha(\alpha-\alpha q))^n \alpha^{j+1}\underline{\alpha} + (\alpha q(\alpha-\alpha q))^n (\alpha q)^{j+1}\underline{\gamma}}{\alpha(1-q)}\right)$$
$$= (\alpha-\alpha q)^n\left[\frac{\alpha^{n+j}\underline{\alpha} + (\alpha q)^{n+j}\underline{\gamma}}{\alpha(1-q)} + \varepsilon\left(\frac{\alpha^{n+j+1}\underline{\alpha} + (\alpha q)^{n+j+1}\underline{\gamma}}{\alpha(1-q)}\right)\right]$$
$$= (\alpha-\alpha q)^{n-1} \widetilde{\mathcal{L}}_{n+j}(\alpha;q).$$

The rest of the results (25), (26) and (27) can be proven analogously. Thus, the proof is completed. $\square$

Theorem 2.5. *For nonnegative integer n, we have*

$$\sum_{k=0}^{n}\binom{n}{2k}\left(-\alpha^2 q\right)^{n-k}\widetilde{\mathcal{F}}_{4k}(\alpha;q)$$
$$= \begin{cases} \frac{1}{2}\Big((\alpha-\alpha q)^n + (\alpha[2]_q)^n\Big)\widetilde{\mathcal{F}}_n(\alpha;q), & \textit{if } n \textit{ is even} \\ \frac{1}{2}\Big((\alpha-\alpha q)^{n-1}\widetilde{\mathcal{L}}_n(\alpha;q) - (\alpha[2]_q)^n\widetilde{\mathcal{F}}_n(\alpha;q)\Big), & \textit{if } n \textit{ is odd} \end{cases}, \quad (29)$$

$$\sum_{k=0}^{n}\binom{n}{2k}\left(-\alpha^{2}q\right)^{n-k}\widetilde{\mathcal{L}}_{4k}(\alpha;q)$$
$$=\begin{cases}\frac{1}{2}\Big(\left(\alpha-\alpha q\right)^{n}+\left(\alpha[2]_{q}\right)^{n}\Big)\widetilde{\mathcal{L}}_{n}(\alpha;q), & \textit{if } n \textit{ is even}\\ \frac{1}{2}\Big(\left(\alpha-\alpha q\right)^{n-1}\widetilde{\mathcal{F}}_{n}(\alpha;q)-\left(\alpha[2]_{q}\right)^{n}\widetilde{\mathcal{L}}_{n}(\alpha;q)\Big), & \textit{if } n \textit{ is odd}\end{cases}, \quad (30)$$

$$\sum_{k=0}^{n}\binom{n}{2k+1}\left(-\alpha^{2}q\right)^{n-k}\widetilde{\mathcal{F}}_{4k+1}(\alpha;q)$$
$$=\begin{cases}\frac{1}{2}\Big(\left(\alpha-\alpha q\right)^{n}-\left(\alpha[2]_{q}\right)^{n}\Big)\widetilde{\mathcal{F}}_{n-1}(\alpha;q), & \textit{if } n \textit{ is even}\\ \frac{1}{2}\Big(\left(\alpha-\alpha q\right)^{n-1}\widetilde{\mathcal{L}}_{n-1}(\alpha;q)+\left(\alpha[2]_{q}\right)^{n}\widetilde{\mathcal{F}}_{n-1}(\alpha;q)\Big), & \textit{if } n \textit{ is odd}\end{cases}, \quad (31)$$

$$\sum_{k=0}^{n}\binom{n}{2k+1}\left(-\alpha^{2}q\right)^{n-k}\widetilde{\mathcal{L}}_{4k+1}(\alpha;q)$$
$$=\begin{cases}\frac{1}{2}\Big(\left(\alpha-\alpha q\right)^{n}-\left(\alpha[2]_{q}\right)^{n}\Big)\widetilde{\mathcal{L}}_{n-1}(\alpha;q), & \textit{if } n \textit{ is even}\\ \frac{1}{2}\Big(\left(\alpha-\alpha q\right)^{n+1}\widetilde{\mathcal{F}}_{n-1}(\alpha;q)+\left(\alpha[2]_{q}\right)^{n}\widetilde{\mathcal{L}}_{n-1}(\alpha;q)\Big), & \textit{if } n \textit{ is odd}\end{cases}. \quad (32)$$

Proof. Firstly, we prove the identity (29). By taking consideration the Theorem 2.4 and using some binomial sum properties, we get

$$\sum_{k=0}^{n}\binom{n}{2k}\left(-\alpha^{2}q\right)^{n-k}\widetilde{\mathcal{F}}_{4k}(\alpha;q)$$
$$=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left(1+(-1)^{k}\right)\left(-\alpha^{2}q\right)^{n-k}\widetilde{\mathcal{F}}_{2k}(\alpha;q)$$
$$=\frac{1}{2}\left[\sum_{k=0}^{n}\binom{n}{k}\left(-\alpha^{2}q\right)^{n-k}\widetilde{\mathcal{F}}_{2k}(\alpha;q)+\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}\left(-\alpha^{2}q\right)^{n-k}\widetilde{\mathcal{F}}_{2k}(\alpha;q)\right]$$
$$=\begin{cases}\frac{1}{2}\Big(\left(\alpha-\alpha q\right)^{n}+\left(\alpha[2]_{q}\right)^{n}\Big)\widetilde{\mathcal{F}}_{n}(\alpha;q), & \text{if } n \text{ is even}\\ \frac{1}{2}\Big(\left(\alpha-\alpha q\right)^{n-1}\widetilde{\mathcal{L}}_{n}(\alpha;q)-\left(\alpha[2]_{q}\right)^{n}\widetilde{\mathcal{F}}_{n}(\alpha;q)\Big), & \text{if } n \text{ is odd}\end{cases}.$$

Thus the proof is completed. By analogy to this proof, the equalities (30), (31) and (32) can be proven. □

Theorem 2.6. *For nonnegative integer n, we have*

$$\sum_{k=0}^{n}\binom{n}{k}\left(\alpha[2]_q\right)^k\left(-\alpha^2 q\right)^{n-k}\widetilde{\mathcal{F}}_k(\alpha;q)=\widetilde{\mathcal{F}}_{2n}(\alpha;q) \tag{33}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\left(\alpha[2]_q\right)^k\left(-\alpha^2 q\right)^{n-k}\widetilde{\mathcal{L}}_k(\alpha;q)=\widetilde{\mathcal{L}}_{2n}(\alpha;q). \tag{34}$$

Proof. First we prove the identity (33). By the help of the Binet formula of the $q-$Fibonacci dual bicomplex numbers in (18), we get

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\left(\alpha[2]_q\right)^k\left(-\alpha^2 q\right)^{n-k}\widetilde{\mathcal{F}}_k(\alpha;q)\\
&=\sum_{k=0}^{n}\binom{n}{k}\left(\alpha(1+q)\right)^k\left(-\alpha^2 q\right)^{n-k}\left[\frac{\alpha^k\underline{\alpha}-(\alpha q)^k\underline{\gamma}}{\alpha(1-q)}+\varepsilon\left(\frac{\alpha^{k+1}\underline{\alpha}-(\alpha q)^{k+1}\underline{\gamma}}{\alpha(1-q)}\right)\right]\\
&=\frac{\underline{\alpha}}{\alpha(1-q)}\sum_{k=0}^{n}\binom{n}{k}\left(\alpha^2(1+q)\right)^k\left(-\alpha^2 q\right)^{n-k}\\
&\qquad-\frac{\underline{\gamma}}{\alpha(1-q)}\sum_{k=0}^{n}\binom{n}{k}\left(\alpha^2 q(1+q)\right)^k\left(-\alpha^2 q\right)^{n-k}\\
&\quad+\varepsilon\left(\frac{\alpha\underline{\alpha}}{\alpha(1-q)}\sum_{k=0}^{n}\binom{n}{k}\left(\alpha^2(1+q)\right)^k\left(-\alpha^2 q\right)^{n-k}\right.\\
&\qquad\qquad\left.-\frac{\alpha q\underline{\gamma}}{\alpha(1-q)}\sum_{k=0}^{n}\binom{n}{k}\left(\alpha^2 q(1+q)\right)^k\left(-\alpha^2 q\right)^{n-k}\right)\\
&=\frac{\alpha^{2n}\underline{\alpha}-(\alpha q)^{2n}\underline{\gamma}}{\alpha(1-q)}+\varepsilon\left(\frac{\alpha^{2n+1}\underline{\alpha}-(\alpha q)^{2n+1}\underline{\gamma}}{\alpha(1-q)}\right)\\
&=\widetilde{\mathcal{F}}_{2n}(\alpha;q).
\end{aligned}$$

On the other hand, the result (34) can be proven in a similar way. □

Theorem 2.7. *For any integers, n, r and s, the $q-$Fibonacci and $q-$Lucas dual bicomplex numbers satisfy the identity:*

$$\begin{aligned}
&\widetilde{\mathcal{L}}_{n+r}(\alpha;q)\widetilde{\mathcal{F}}_{n+s}(\alpha;q)-\widetilde{\mathcal{L}}_{n+s}(\alpha;q)\widetilde{\mathcal{F}}_{n+r}(\alpha;q)\\
&\qquad=\frac{\alpha^{2n+r+s-1}q^n\left(\alpha[2]_q\varepsilon+1\right)\left(q^r-q^s\right)\left(\underline{\alpha}\underline{\gamma}+\underline{\gamma}\underline{\alpha}\right)}{1-q}.
\end{aligned}$$

Proof. By virtue of both (18) and (19), we have

$$\widetilde{\mathcal{L}}_{n+r}(\alpha;q)\widetilde{\mathcal{F}}_{n+s}(\alpha;q)-\widetilde{\mathcal{L}}_{n+s}(\alpha;q)\widetilde{\mathcal{F}}_{n+r}(\alpha;q)$$

$$\begin{aligned}
&=\left(\alpha^{n+r}\underline{\alpha}+(\alpha q)^{n+r}\underline{\gamma}+\varepsilon\left(\alpha^{n+r+1}\underline{\alpha}+(\alpha q)^{n+r+1}\underline{\gamma}\right)\right)\\
&\quad\times\left(\frac{\alpha^{n+s}\underline{\alpha}-(\alpha q)^{n+s}\underline{\gamma}}{\alpha(1-q)}+\varepsilon\left(\frac{\alpha^{n+s+1}\underline{\alpha}-(\alpha q)^{n+s+1}\underline{\gamma}}{\alpha(1-q)}\right)\right)\\
&\quad-\left(\alpha^{n+s}\underline{\alpha}+(\alpha q)^{n+s}\underline{\gamma}+\varepsilon\left(\alpha^{n+s+1}\underline{\alpha}+(\alpha q)^{n+s+1}\underline{\gamma}\right)\right)\\
&\quad\times\left(\frac{\alpha^{n+r}\underline{\alpha}-(\alpha q)^{n+r}\underline{\gamma}}{\alpha(1-q)}+\varepsilon\left(\frac{\alpha^{n+r+1}\underline{\alpha}-(\alpha q)^{n+r+1}\underline{\gamma}}{\alpha(1-q)}\right)\right).
\end{aligned}$$

After some calculus, we get

$$\begin{aligned}
&\widetilde{\mathcal{L}}_{n+r}(\alpha;q)\widetilde{\mathcal{F}}_{n+s}(\alpha;q)-\widetilde{\mathcal{L}}_{n+s}(\alpha;q)\widetilde{\mathcal{F}}_{n+r}(\alpha;q)\\
&\qquad=\frac{\alpha^{2n+r+s-1}q^{n}(\alpha[2]_q\varepsilon+1)\left(q^{r}-q^{s}\right)\left(\underline{\alpha}\underline{\gamma}+\underline{\gamma}\underline{\alpha}\right)}{1-q}.
\end{aligned}$$

□

Theorem 2.8 (Catalan's identity)**.** *For positive integers n and r such that $n \geq r$, we have*

$$\widetilde{\mathcal{F}}_{n+r}(\alpha;q)\widetilde{\mathcal{F}}_{n-r}(\alpha;q)-\widetilde{\mathcal{F}}^{2}_{n}(\alpha;q)=\frac{\alpha^{2n-2}q^{n-r}\left(q^{r}-1\right)\left(\underline{\gamma}\underline{\alpha}q^{r}-\underline{\alpha}\underline{\gamma}\right)\left(1+\alpha[2]_q\varepsilon\right)}{(1-q)^{2}} \tag{35}$$

and

$$\widetilde{\mathcal{L}}_{n+r}(\alpha;q)\widetilde{\mathcal{L}}_{n-r}(\alpha;q)-\widetilde{\mathcal{L}}^{2}_{n}(\alpha;q)=\alpha^{2n}q^{n-r}\left(q^{r}-1\right)\left(\underline{\gamma}\underline{\alpha}q^{r}-\underline{\alpha}\underline{\gamma}\right)\left(1+\alpha[2]_q\varepsilon\right). \tag{36}$$

Proof. Using the Binet formula of the $q-$Fibonacci dual bicomplex numbers in (18), we find that

$$\widetilde{\mathcal{F}}_{n+r}(\alpha;q)\widetilde{\mathcal{F}}_{n-r}(\alpha;q)-\widetilde{\mathcal{F}}^{2}_{n}(\alpha;q)$$

$$
\begin{aligned}
&= \left(\frac{\alpha^{n+r}\underline{\alpha} - (\alpha q)^{n+r}\underline{\gamma}}{\alpha(1-q)} + \varepsilon \left(\frac{\alpha^{n+r+1}\underline{\alpha} - (\alpha q)^{n+r+1}\underline{\gamma}}{\alpha(1-q)} \right) \right) \\
&\quad \times \left(\frac{\alpha^{n-r}\underline{\alpha} - (\alpha q)^{n-r}\underline{\gamma}}{\alpha(1-q)} + \varepsilon \left(\frac{\alpha^{n-r+1}\underline{\alpha} - (\alpha q)^{n-r+1}\underline{\gamma}}{\alpha(1-q)} \right) \right) \\
&\quad - \left(\frac{\alpha^{n}\underline{\alpha} - (\alpha q)^{n}\underline{\gamma}}{\alpha(1-q)} + \varepsilon \left(\frac{\alpha^{n+1}\underline{\alpha} - (\alpha q)^{n+1}\underline{\gamma}}{\alpha(1-q)} \right) \right)^2 .
\end{aligned}
$$

After some calculations, we arrive at the desired result:

$$
\widetilde{\mathcal{F}}_{n+r}(\alpha;q)\widetilde{\mathcal{F}}_{n-r}(\alpha;q) - \widetilde{\mathcal{F}}_{n}^{2}(\alpha;q) = \frac{\alpha^{2n-2}q^{n-r}\left(q^r-1\right)\left(\underline{\gamma}\underline{\alpha}q^r - \underline{\alpha}\underline{\gamma}\right)\left(1+\alpha[2]_q\varepsilon\right)}{(1-q)^2}.
$$

Furthermore, from (19), the equality (36) can be proven in a similar way. So, the proof is completed. □

Theorem 2.9 (Cassini's identity)**.** *For $n \geq 1$, the following identities hold:*

$$
\widetilde{\mathcal{F}}_{n+1}(\alpha;q)\widetilde{\mathcal{F}}_{n-1}(\alpha;q) - \widetilde{\mathcal{F}}_{n}^{2}(\alpha;q) = \frac{\alpha^{2n-2}q^{n-1}\left(\underline{\alpha}\underline{\gamma} - \underline{\gamma}\underline{\alpha}q\right)\left(1+\alpha[2]_q\varepsilon\right)}{1-q} \tag{37}
$$

and

$$
\widetilde{\mathcal{L}}_{n+1}(\alpha;q)\widetilde{\mathcal{L}}_{n-1}(\alpha;q) - \widetilde{\mathcal{L}}_{n}^{2}(\alpha;q) = \alpha^{2n}q^{n-1}\left(q-1\right)\left(\underline{\gamma}\underline{\alpha}q - \underline{\alpha}\underline{\gamma}\right)\left(1+\alpha[2]_q\varepsilon\right). \tag{38}
$$

Proof. As this identity is the special case of the Theorem 2.8 for $r = 1$, the proof is trivial. □

Theorem 2.10 (d'Ocagne's identity)**.** *Let n be a nonnegative integer and m be a natural number. If $m > n+1$, then we have*

$$
\begin{aligned}
&\widetilde{\mathcal{F}}_{m}(\alpha;q)\widetilde{\mathcal{F}}_{n+1}(\alpha;q) - \widetilde{\mathcal{F}}_{n}(\alpha;q)\widetilde{\mathcal{F}}_{m+1}(\alpha;q) \\
&\qquad = \frac{\alpha^{m+n-1}\left(q^m - q^n\right)\left(\underline{\alpha}\underline{\gamma}q - \underline{\gamma}\underline{\alpha}\right)(1+\alpha[2]_q\varepsilon)}{(1-q)^2}
\end{aligned} \tag{39}
$$

and

$$
\begin{aligned}
&\widetilde{\mathcal{L}}_{m}(\alpha;q)\widetilde{\mathcal{L}}_{n+1}(\alpha;q) - \widetilde{\mathcal{L}}_{n}(\alpha;q)\widetilde{\mathcal{L}}_{m+1}(\alpha;q) \\
&\qquad = \alpha^{m+n+1}\left(q^m - q^n\right)\left(\underline{\gamma}\underline{\alpha} - \underline{\alpha}\underline{\gamma}q\right)(1+\alpha[2]_q\varepsilon).
\end{aligned} \tag{40}
$$

Proof. By means of Binet formula of the $q-$Fibonacci dual bicomplex numbers in (18), we obtain

$$\widetilde{\mathcal{F}}_m(\alpha;q)\widetilde{\mathcal{F}}_{n+1}(\alpha;q)-\widetilde{\mathcal{F}}_n(\alpha;q)\widetilde{\mathcal{F}}_{m+1}(\alpha;q)$$

$$\begin{aligned}
&=\left(\frac{\alpha^m\underline{\alpha}-(\alpha q)^m\underline{\gamma}}{\alpha(1-q)}+\varepsilon\left(\frac{\alpha^{m+1}\underline{\alpha}-(\alpha q)^{m+1}\underline{\gamma}}{\alpha(1-q)}\right)\right)\\
&\quad\times\left(\frac{\alpha^{n+1}\underline{\alpha}-(\alpha q)^{n+1}\underline{\gamma}}{\alpha(1-q)}+\varepsilon\left(\frac{\alpha^{n+2}\underline{\alpha}-(\alpha q)^{n+2}\underline{\gamma}}{\alpha(1-q)}\right)\right)\\
&\quad-\left(\frac{\alpha^n\underline{\alpha}-(\alpha q)^n\underline{\gamma}}{\alpha(1-q)}+\varepsilon\left(\frac{\alpha^{n+1}\underline{\alpha}-(\alpha q)^{n+1}\underline{\gamma}}{\alpha(1-q)}\right)\right)\\
&\quad\times\left(\frac{\alpha^{m+1}\underline{\alpha}-(\alpha q)^{m+1}\underline{\gamma}}{\alpha(1-q)}+\varepsilon\left(\frac{\alpha^{m+2}\underline{\alpha}-(\alpha q)^{m+2}\underline{\gamma}}{\alpha(1-q)}\right)\right).
\end{aligned}$$

After some calculations, we get

$$\begin{aligned}
&\widetilde{\mathcal{F}}_m(\alpha;q)\widetilde{\mathcal{F}}_{n+1}(\alpha;q)-\widetilde{\mathcal{F}}_n(\alpha;q)\widetilde{\mathcal{F}}_{m+1}(\alpha;q)\\
&\qquad=\frac{\alpha^{m+n-1}\left(q^m-q^n\right)\left(\underline{\alpha}\underline{\gamma}q-\underline{\gamma}\underline{\alpha}\right)(1+\alpha[2]_q\varepsilon)}{(1-q)^2}. \qquad (41)
\end{aligned}$$

In a similar way, from (19), the equality (40) can be derived. So, the proof is completed. □

3 Conclusion

In this study, we introduce two family of dual bicomplex numbers with components containing $q-$integers. First, we define $q-$Fibonacci and $q-$Lucas dual bicomplex numbers. We give several algebraic properties, exponential generating functions and the binomial sums. Besides, we touch upon that these numbers can be reduced into several new dual bicomplex numbers for the special cases of q and α. Afterwards, by means of the Binet formula of these numbers, we investigate several identities such as Catalan' identity, Cassini's identity, d'Ocagne's identity and a general identity (see Theorem 2.7). Thus, this study can be described as a study involving the connection between dual bicomplex numbers and $q-$calculus.

On the other hand, in [7] and [4], the authors defined the biperiodic Fibonacci and the biperiodic Lucas sequences. By virtue of these sequences, it would be interesting to study $q-$analog of the dual bicomplex biperiodic Fibonacci and Lucas numbers.

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